14. The mean value and maximum property

Definition 14.1. *Let $h\colon U \longrightarrow \mathbb{R}$ be a continuous, real-valued function on a region U. Suppose that the open disk of radius ρ centred around a is contained in U.*

*The **average value of** h on the circle $|z-a| = r$ is defined to be*

$$A(r) = \frac{1}{2\pi}\int_0^{2\pi} h(a + re^{i\theta})\,\mathrm{d}\theta \qquad \textit{where} \qquad r \in (0, \rho).$$

As h is a continuous function it follows that the average value $A(r)$ is a continuous function of r. The average of h at a is presumably $h(a)$ and so one might guess that $A(r)$ approaches $h(a)$ as r approaches zero. In fact this is the case:

Lemma 14.2. *Let $h\colon U \longrightarrow \mathbb{R}$ be a continuous, real-valued function on a region U.*

Then

$$\lim_{r\to 0} A(r) = h(a) \qquad \textit{where} \qquad A(r) = \frac{1}{2\pi}\int_0^{2\pi} h(a + re^{i\theta})\,\mathrm{d}\theta$$

is the average value.

Proof. We have

$$\begin{aligned}
|A(r) - h(a)| &= \left| \frac{1}{2\pi}\int_0^{2\pi} h(a + re^{i\theta}) - h(a)|\,\mathrm{d}\theta \right| \\
&= \frac{1}{2\pi}\int_0^{2\pi} |h(a + re^{i\theta}) - h(a)|\,\mathrm{d}\theta \\
&\leq M,
\end{aligned}$$

where $M = M(r)$ is the maximum value of $|h(a + re^{i\theta}) - h(a)|$. As h is continuous, M approaches zero as r approaches 0. □

Theorem 14.3. *If u is a harmonic function on a region U and if the open disk of radius ρ centred at a is contained in U then*

$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(a + re^{i\theta})\,\mathrm{d}\theta \qquad \textit{where} \qquad r \in (0, \rho).$$

Proof. We give two proofs.

For the first proof, note that

$$\begin{aligned}
\oint_{|z-a|=r} -\frac{\partial u}{\partial y}\,\mathrm{d}x + \frac{\partial u}{\partial x}\,\mathrm{d}y &= \iint_{|z-a|<r} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \mathrm{d}x\mathrm{d}y \\
&= \iint_{|z-a|<r} 0\,\mathrm{d}x\mathrm{d}y \\
&= 0.
\end{aligned}$$

The first line follows from Green's theorem. We parametrise the circle by

$$x(\theta) = x_0 + r\cos\theta \qquad \text{and} \qquad y(\theta) = y_0 + r\sin\theta.$$

It follows that

$$\begin{aligned}
0 &= \oint_{|z-a|=r} -\frac{\partial u}{\partial y}\,\mathrm{d}x + \frac{\partial u}{\partial x}\,\mathrm{d}y \\
&= r\int_0^{2\pi} \left[\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right] \mathrm{d}\theta \\
&= r\int_0^{2\pi} \frac{\partial u}{\partial r}(a + re^{i\theta})\,\mathrm{d}\theta.
\end{aligned}$$

Dividing by $2\pi r$ and taking the derivative out of the integral sign gives

$$\frac{\partial}{\partial r}\frac{1}{2\pi}\int_0^{2\pi} u(a + re^{i\theta})\,\mathrm{d}\theta = 0.$$

It follows that $A(r)$ is constant. As it approaches $u(a)$ as r goes to zero, we must have

$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(a + re^{i\theta})\,\mathrm{d}\theta.$$

For the second proof, pick a harmonic conjugate v and let $f = u + iv$. Then f is holomorphic. Cauchy's integral formula implies that

$$f(a) = \frac{1}{2\pi i}\oint_{|z-a|=r} \frac{f(z)}{z-a}\,\mathrm{d}z.$$

Suppose that we parametrise the circle

$$z = a + re^{i\theta} \qquad \text{so that} \qquad \frac{\mathrm{d}z}{i(z-a)} = \mathrm{d}\theta.$$

It follows that

$$f(a) = \frac{1}{2\pi}\oint_0^{2\pi} f(a + re^{i\theta})\,\mathrm{d}\theta.$$

Now take real parts. $\square$

The property that the average value of u around a circle is equal to the value at the centre is called the *mean value property*. We have just established that harmonic functions satisfy the mean value property. It is an amazing fact that any continuous function which satisfies the mean value property is in fact harmonic.

Theorem 14.4 (Strict maximum principle)**.** *Let u be a harmonic function on a region U such that $u(z) \leq M$ on U.*

If there is a point $a \in U$ such that $u(a) = M$ then u is constant.

Proof. Let $V \subset U$ be the set of points $b \in U$ such that $u(b) = M$. By assumption V is non-empty as $a \in V$.

Suppose that $b \in V$. As U is open we can find an open disk centred at b contained in U. We have

$$u(b) = \frac{1}{2\pi} \int_0^{2\pi} u(b + re^{i\theta}) \, \mathrm{d}\theta,$$

for any sufficiently small r. But then

$$0 = \frac{1}{2\pi} \int_0^{2\pi} u(b) - u(b + re^{i\theta}) \, \mathrm{d}\theta,$$

The integrand

$$u(b) - u(b + re^{i\theta}) \geq 0,$$

by assumption. As the area under the graph is zero the only possibility is that

$$u(b) - u(b + re^{i\theta}) = 0$$

all the way around the boundary.

It follows that every point c whose distance to b is r belongs to V. Varying r we see that an open disk centred around b is contained in V. It follows that V is open and so $V = U$ as U is connected. $\square$

We can extend the maximum principle for harmonic functions to holomorphic functions

Theorem 14.5 (Strict maximum principle)**.** *Let f be a holomorphic function on a region U such that $|f(z)| \leq M$ on U.*

If there is a point $a \in U$ such that $|f(a)| = M$ then u is constant.

Proof. Suppose that

$$f(a) = Me^{i\theta}.$$

Let

$$g \colon U \longrightarrow \mathbb{C}$$

be the function

$$z \longrightarrow e^{-i\theta} f(z)$$

Then

$$|g(z)| = |f(z)| \qquad \text{and} \qquad g(a) = M.$$

It follows that $|g(z)| \leq M$ on U.

Let u be the real part of $g = u + iv$. Then u is a harmonic function on U,

$$\begin{aligned} u(z) &\leq |g(z)| \\ &\leq M \end{aligned}$$

and

$$u(a) = M.$$

Thus $u(z)$ is constant by (14.4). Thus $u(z) = M$. As $|g(z)| \leq M$ it follows that $v = 0$ and $g(z) = M$ is constant. But then $f(z)$ is constant. □

Theorem 14.6 (Maximum principle)**.** *Let $f(z)$ be a holomorphic function on a bounded region U which extends to a continuous funcion on the boundary.*

If $|f(z)| \leq M$ on the boundary ∂U then $|f(z)| \leq M$ on the whole of U.

Proof. The follows from a basic fact in topology that the continuous function $|f(z)|$ must attain its maximum at some point of $U \cup \partial U$. If it is on U then the strong maximum principle implies that f is constant and the result is clear.

If $|f(z)|$ attains its maximum on the boundary of U then the result is also clear. □