13. Harmonic functions

Definition 13.1. *Let $U \subset \mathbb{C}$ be a region in the plane. Let*

$$u\colon U \longrightarrow \mathbb{R}$$

be a real valued function on U with continuous 2nd order partial derivatives.

We say that u is ***harmonic*** *if u satisfies* ***Laplace's equation****:*

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Laplace's equation is one of the most important partial differential equations of mathematical physics.

Theorem 13.2. *If $f\colon U \longrightarrow \mathbb{C}$ is a holomorphic function on a region U and $f = u + iv$ then u and v are harmonic functions.*

Proof. As f is holomorphic, it is infinitely differentiable. In particular u and v have continuous 2nd order partial derivatives.

u and v satisfy the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

It follows that

$$\begin{aligned}\Delta u &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \\ &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) - \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) \\ &= 0.\end{aligned}$$

Note that to get from the 3rd line to the 4th line we used the fact that the 2nd partial derivatives of v are continuous to conclude that the mixed partials are equal. □

Definition 13.3. *Let $u\colon U \longrightarrow \mathbb{R}$ be a harmonic function on a region U.*

A harmonic function $v\colon U \longrightarrow \mathbb{R}$ is called a ***harmonic conjugate*** *of u if $f = u + iv$ is holomorphic.*

Note that if v is a harmonic conjugate of u and a is a complex number then $v + a$ is also a harmonic conjugate, as

$$\begin{aligned} u + i(v + a) &= u + iv + ia \\ &= f + ia, \end{aligned}$$

is holomorphic. Conversely, if v and w are two harmonic conjugates of u then

$$\begin{aligned} i(w - v) &= (u + iv) - (u + iw) \\ &= f - g, \end{aligned}$$

is holomorphic, as it is the difference of two holomorphic functions. As $i(w - v)$ is purely imaginary and holomorphic, it must be constant. Thus $w = v + a$, for some complex number a.

Thus harmonic conjugates are unique up to adding a constant.

Example 13.4. *Show that $u = xy$ is harmonic on the whole complex plane and find a harmonic conjugate.*

It is clear that u has continuous 2nd partial derivatives. We have

$$\frac{\partial u}{\partial x} = y \qquad \text{and} \qquad \frac{\partial u}{\partial y} = x.$$

It follows that

$$\begin{aligned} \Delta u &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \\ &= 0 + 0 \\ &= 0. \end{aligned}$$

Thus u is harmonic.

To find a harmonic conjugate, we solve the Cauchy-Riemann equations:

$$v_y = y \qquad \text{and} \qquad v_x = -x.$$

To solve the first partial differential equation we integrate both sides with respect to y. Thus

$$v(x, y) = \frac{y^2}{2} + h(x),$$

where $h(x)$ is an arbitrary function of x. If we plug this into the second equation then we get

$$h'(x) = -x.$$

Thus

$$h(x) = -\frac{x^2}{2}.$$

Hence

$$v(x,y) = \frac{y^2}{2} - \frac{x^2}{2}.$$

Note that in fact

$$f(z) = -i\frac{z^2}{2}$$

has real part u and imaginary part v.

The same method show that we can find the harmonic conjugate v of any harmonic function u on a rectangle U whose sides are parallel to the axes.

Indeed pick a point $(x_0, y_0) \in U$. We start by integrating the first Cauchy-Riemann equation with respect to y to get

$$v(x,y) = \int_{y_0}^{y} \frac{\partial u}{\partial x}(x,t)\,\mathrm{d}t + h(x).$$

Here $h(x)$ is a constant of integration. Since we integrated with respect to y, this constant does not depend on y but it does depend on x.

If we plug this value into the 2nd Cauchy-Riemann equation then we get

$$\begin{aligned}
\frac{\partial u}{\partial y}(x,y) &= -\frac{\partial}{\partial x}\int_{y_0}^{y} \frac{\partial u}{\partial x}(x,t)\,\mathrm{d}t - h'(x) \\
&= -\int_{y_0}^{y} \frac{\partial^2 u}{\partial x^2}(x,t)\,\mathrm{d}t - h'(x) \\
&= \int_{y_0}^{y} \frac{\partial^2 u}{\partial y^2}(x,t)\,\mathrm{d}t - h'(x) \\
&= \frac{\partial u}{\partial y}(x,y) - \frac{\partial u}{\partial y}(x,y_0) - h'(x).
\end{aligned}$$

Hence

$$h'(x) = -\frac{\partial u}{\partial y}(x,y_0).$$

Integrating both sides with respect to x we get

$$h(x) = -\int_{x_0}^{x} \frac{\partial u}{\partial y}(s,y_0)\,\mathrm{d}s,$$

up to a constant.

Thus

$$v(x,y) = \int_{y_0}^{y} \frac{\partial u}{\partial x}(x,t)\,\mathrm{d}t - \int_{x_0}^{x} \frac{\partial u}{\partial y}(s,y_0)\,\mathrm{d}s,$$

up to a constant.

Theorem 13.5. *Let U be an open disk or an open rectangle with sides parallel to the axes.*

Then every harmonic function on U has a harmonic conjugate.

Proof. We already saw a proof of this result for rectangles.

If U is an open disk then we use the same proof, except that we have no choice but to put (x_0, y_0) at the centre of the circle. This way, given any point (x, y) in the open disk, we can get to (x, y) by first going from x_0 up to x, so that we go from (x_0, y_0) to (x, y_0) and then go from y_0 to y, so that we go from (x, y_0) to (x, y).

This unambiguously defines a function $v(x, y)$ by integration, see the formula above. It is then clear that v is a harmonic conjugate of u. □