12. Rouches Theorem

Looking at how we used the argument principle in lecture 11, it is hopefully clear that it is often very useful to identify the dominant term and exploit the fact that this controls the change in the argument.

It is also very useful to have a general purpose result. We start with a reformulation of (11.2) that we have been already using implicitly:

Theorem 12.1. *Let U be a bounded region whose boundary $\gamma = \partial U$ is piecewise smooth.*

If f is a meromorphic function on U that is holomorphic and non-zero on ∂U then the change in the argument divided by 2π is equal to

$$N_0 - N_\infty.$$

Theorem 12.2 (Rouché's Theorem)**.** *Let U be a bounded region with piecewise smooth boundary ∂U. Suppose that $f(z)$ and $h(z)$ are holomorphic on $U \cup \partial U$.*

If $|h(z)| < |f(z)|$ on ∂U then $f(z)$ and $f(z) + g(z)$ have the same number of zeroes in U, counting multiplicities.

Proof. Note that $f(z)$ is nowhere zero on ∂U,

$$|f(z)| > |h(z)| > 0.$$

Similarly $f(z) + h(z)$ is nowhere zero on ∂U, as

$$\begin{aligned} |f(z) + h(z)| &\geq |f(z)| - |h(z)| \\ &> 0. \end{aligned}$$

As

$$f(z) + h(z) = f(z)\left[1 + \frac{h(z)}{f(z)}\right]$$

it follows that

$$\arg(f(z) + h(z)) = \arg(f(z)) + \arg\left(1 + \frac{h(z)}{f(z)}\right).$$

As

$$\left|\frac{h(z)}{f(z)}\right| < 1$$

it follows that

$$1 + \frac{h(z)}{f(z)}$$

always has positive real part, so that this complex number is always in the right half plane $\mathrm{Re}(z) > 0$. It follows that the argument varies continuously between $-\pi/2$ and $\pi/2$. In particular the change in the argument around a closed curve is zero, since the change lies in the interval $(-\pi, \pi)$ and it is a multiple of 2π.

Thus the change in the argument of $f(z)+h(z)$ is equal to the change in the argument of $f(z)$ around the boundary of U. Thus $f(z)$ and $f(z)+h(z)$ have the same number of zeroes, counting multiplicities. $\square$

Example 12.3. *How many zeroes does*

$$p(z) = z^6 + 9z^4 + z^3 + 2z + 4$$

have in the unit disk?

Let $U = \Delta$. Then the boundary of U is the unit circle. To apply (12.2), our goal is to write $p(z)$ as the sum of two functions, one of which is bigger than the other on the unit circle. Note that

$$|z^m| = 1$$

on the unit circle, regardless of m, so that the power of z is invisible. The biggest term is then

$$f(z) = 9z^4.$$

We have to compare this with what is left, namely

$$h(z) = z^6 + z^3 + 2z + 4.$$

On the unit circle

$$\begin{aligned}
|h(z)| &= |z^6 + z^3 + 2z + 4| \\
&\leq |z^6| + |z^3| + |2z| + |4| \\
&= 1 + 1 + 2 + 4 \\
&= 8 \\
&\leq 9 \\
&= |9z^4| \\
&= |f(z)|.
\end{aligned}$$

Thus

$$p(z) = f(z) + h(z)$$

has the same number of zeroes as $f(z) = 9z^4$ on the unit disk. But $f(z)$ has four zeroes and so

$$p(z) = z^6 + 9z^4 + z^3 + 2z + 4$$

has four zeroes on the unit disk.

Example 12.4. *Solve*

$$e^z = 1 + 2z \qquad \textit{subject to} \qquad |z| < 1.$$

One obvious solution is $z = 0$. Let

$$p(z) = e^z - 1 - 2z.$$

Then we want to find the zeroes of $p(z)$. On the the unit circle the dominant term is $f(z) = -2z$ and then the remaining part is $h(z) = e^z - 1$.

We check the hypothesis of (12.2). It is clear that both $f(z)$ and $h(z)$ are holomorphic on the closed unit disk. On the unit circle we have

$$|f(z)| = 2$$

and

$$\begin{aligned} |h(z)| &= |e^z - 1| \\ &= \left| z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots \right| \\ &\leq |z| + \frac{|z|^2}{2!} + \frac{|z|^3}{3!} + \dots \\ &= 1 + \frac{1}{2!} + \frac{1}{3!} + \dots \\ &= e - 1 \\ &< 2. \end{aligned}$$

Thus (12.2) implies that $p(z)$ has the same number of zeroes as $f(z) = -2z$ in the unit disk. As $f(z)$ has one zero at 0, it follows that $p(z)$ has one zero.

So the only solution of the equation

$$e^z = 1 + 2z \qquad \text{subject to} \qquad |z| < 1.$$

is $z = 0$.

Corollary 12.5. *Every non-constant polynomial with complex roots has a root.*

Proof. Suppose that

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0,$$

where $a_0, a_1, \dots, a_n$ are complex numbers, $n > 0$ and $a_n \neq 0$.

We may assume that $a_n = 1$. We consider what happens in a large open disk centred at the origin of radius R. On the circle $|z| = R$ the dominant term is $f(z) = z^n$. Let

$$h(z) = p(z) - f(z) = +a_{n-1} z^{n-1} + \dots + a_1 z + a_0.$$

We have

$$|h(z)| < |f(z)|$$

on the circle $|z| = R$, so that $p(z)$ has the same number of zeroes on the open disk $|z| < R$, as $f(z) = z^n$, which is zero at the origin. $\square$