11. The argument principle

Definition 11.1. *Let $f\colon U \longrightarrow \mathbb{C}$ be a function on a region U.*

*We say that f is **meromorphic** on U if it has only isolated singularities on U, all of which are poles.*

*The **order of a meromorphic function** at a point a, denoted $\operatorname{ord}_a f(z)$, is the order of zero of $f(z)$ at a, if $f(z)$ is holomorphic at a and otherwise it is minus the order of the pole of $f(z)$ at a.*

N_0 is the sum of the order of all of the zeroes and N_∞ is the sum of all of the orders of the poles.

N_0 counts the number of zeroes, according to multiplicity and N_∞ counts the number of poles, according to multiplicity.

If one of N_0 and N_∞ is finite then $N_0 - N_\infty$ is the sum of the order of all points of U (most points of U have zero order and make no contribution to the sum).

Let $f\colon U \longrightarrow \mathbb{C}$ be a meromorphic function on a region U. Let γ be a piecewise differentiable path in U such that $f(z)$ is holomorphic and $f(z) \neq 0$ on γ. The integral

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z = \frac{1}{2\pi i}\int_\gamma \mathrm{d}\log f(z)$$

is called the **logarithmic integral** of $f(z)$ along γ.

The logarithmic integral measures the change in the logarithm along a path. This quantity is suprisingly useful:

Theorem 11.2. *Let U be a bounded region whose boundary $\gamma = \partial U$ is piecewise smooth.*

If f is a meromorphic function on U that is holomorphic and non-zero on ∂U then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z = N_0 - N_\infty.$$

Proof. Since $f(z)$ is holomorphic on ∂U, it has finitely poles inside U and since it is non-zero on ∂U it has finitely many zeroes.

Consider

$$g(z) = \frac{f'(z)}{f(z)}.$$

Then the only singularities of $g(z)$ are where $f(z)$ has singularities or where $f(z)$ is zero. Thus $g(z)$ has finitely many isolated singularities in U and $g(z)$ is holomorphic on ∂U.

Therefore we may apply the Residue Theorem to $g(z)$:

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z = \sum_a \operatorname{Res}_a g(z).$$

Pick a point $a \in U$. We may write

$$f(z) = (z-a)^n h(z)$$

where n is the order of a and $h(z)$ is holomorphic at a and non-zero at a. Note that

$$f'(z) = n(z-a)^{n-1}h(z) + (z-a)^n h'(z).$$

It follows that

$$\begin{aligned} g(z) &= \frac{f'(z)}{f(z)} \\ &= \frac{n(z-a)^{n-1}h(z) + (z-a)^n h'(z)}{(z-a)^n h(z)} \\ &= \frac{nh(z) + (z-a)h'(z)}{(z-a)h(z)} \\ &= \frac{n}{z-a} + \frac{h'(z)}{h(z)}. \end{aligned}$$

Note that the second term is holomorphic at a. We can now compute the residue at a:

$$\operatorname{Res}_a g(z) = n.$$

Putting this back into the residue theorem we get

$$\begin{aligned} \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z &= \sum_a \operatorname{Res}_a g(z) \\ &= \sum_a \operatorname{ord}_a f(z) \\ &= N_0 - N_\infty. \end{aligned}$$

□

Let us look more carefully at the logarithmic integral. As

$$\log f(z) = \ln|f(z)| + i\arg f(z)$$

we have

$$\mathrm{d}\log f(z) = \mathrm{d}\ln|f(z)| + \mathrm{d}\arg f(z).$$

Therefore

$$\frac{1}{2\pi i}\int_\gamma \mathrm{d}\log f(z) = \frac{1}{2\pi i}\int_\gamma \mathrm{d}\ln|f(z)| + \frac{1}{2\pi}\int_\gamma \mathrm{d}\arg f(z).$$

Now the first integral on the RHS is not so complicated. As there is no ambiguity in the definition of $\ln|f(z)|$ it follows that

$$\begin{aligned} \int_\gamma \mathrm{d}\ln|f(z)| &= \ln|f(\beta)| - \ln|f(\alpha)| \\ &= \ln|f(b)| - \ln|f(a)|, \end{aligned}$$

where

$$\gamma \colon [\alpha, \beta] \longrightarrow U$$

and $a = \gamma(\alpha)$, $b = \gamma(\beta)$ are the endpoints of γ. The key point is that this integral does not depend on the curve γ only the endpoints.

The second integral is much more subtle, since the value of the integral depends not only on the endpoints but also on the way to get from a to b. The second integral measures the **change in the argument** along the curve $f \circ \gamma$.

If γ is closed curve, that is, $a = b$ then the first integral disappears and the second integral is an integer, equal to $N_0 - N_\infty$.

Example 11.3. *Calculate the change in the argument for the function*

$$z \longrightarrow z^2$$

if we go around the unit circle.

It is natural to divide the unit circle into two pieces, the bit in the upper half plane and the piece in the lower half. If we go around the top, from 1 to -1 the change in the argument for z is π but for z^2 is 2π. Now when we go around the lower half plane, the argument for z goes from π to 2π and the change is π. For z^2 we go from 2π to 4π (or we go from 0 to 2π, we only care about the change). The change is again to 2π.

In total then the change in the argument is 4π. If we divide by 2π then we get 2, accounting for the fact that we go twice around the circle.

Example 11.4. *Calculate the change in the argument for the function*

$$z \longrightarrow z^n$$

if we go around the unit circle.

Suppose first that $n > 0$. Now we divide the circle into n pieces, from 1 to $e^{2\pi i/n}$, and so on. Over each piece the change in the argument is 2π. In total the change in the argument is $2\pi n$. If we divide by 2π we get n, representing the fact that we go around the unit circle n times.

In fact we could have calculated the change in the argument by using (11.2). The function $z \longrightarrow z^2$ is holomorphic on the closed unit disc. It has one zero at $z = 0$ but the order is 2 and so

$$N_0 - N_\infty = 2 - 0,$$

the number of times we go around the circle. For the function $z \longrightarrow z^n$ we have a zero of order n.

Now suppose $n < 0$. As before we divide the circle into n pieces but now we go around the circle backwards, clockwise, instead of anticlockwise. So the change in the argument is now -2π on each piece, making a total of $-2\pi n$. Dividing by 2π we go around the circle n times, but clockwise, accounting for the minus sign. This is again consistent with (11.2). Indeed $z \longrightarrow z^n$ now has no zeroes and one pole at the origin, of order n. We have

$$N_0 - N_\infty = 0 - n = -n.$$

Example 11.5. *How many roots does the polynomial*

$$p(z) = z^6 + 9z^4 + z^3 + 2z + 4$$

have in the first and in the second quadrants?

Consider

$$p(x) = x^6 + 9x^4 + x^3 + 2x + 4.$$

If $x \geq 0$ then this is positive. If $x \in [-1, 0]$ then

$$4 + 2x + x^3 > 0 \qquad \text{so that} \qquad p(x) > 0.$$

If $x \leq -1$ then

$$9x^4 + x^3 + 2x > 0 \qquad \text{so that} \qquad p(x) > 0.$$

Thus $p(z)$ has no real zeroes.

As $p(z)$ has real coefficients it roots come in complex conjugate pairs. Thus three roots lie in the upper half plane and three roots lie in the lower half plane.

Let U be the intersection of the open disk of radius R centred around the origin lying in the first quadrant. We now estimate the change in the argument if we go around the boundary of U, when R is large.

We break the boundary into three pieces,

$$\gamma = \gamma_1 + \gamma_2 + \gamma_3.$$

γ_1 is the straight line from 0 to R along the real axis. As $p(z)$ has real coefficients $p(z)$ is real along γ_1 and there is no change in the argument. γ_2 is the arc of the circle from 0 to iR. If R is sufficiently large then the dominant term is z^6 and so the change in the argument is approximately

$$6\frac{\pi}{2} = 3\pi.$$

γ_3 is is the vertical line segment from iR down to 0. We use the parametrisation

$$\gamma_3(y) = iy \qquad \text{where} \qquad y \in [0, R].$$

It follows that

$$p(iy) = -y^6 + 9y^4 + 4 + i(-y^3 + 2y).$$

When $y = R$ the dominant term is

$$-y^6 = -R^6 < 0.$$

The dominant term for the imaginary part of $p(iy)$ is

$$-y^3 = -R^3 < 0.$$

So at iR the argument is approximately π and $p(iR)$ belongs to the third quadrant.

When y is close to zero $p(iy)$ is close to 4. The dominant imaginary term is $2y$ and so we approach 4 from above, in the first quadrant. The argument is zero at the end.

The final thing is to decide how we got from the third quadrant to 4. We just have to keep track of where we cross the real line. This is when the imaginary part is zero, that is, when

$$y^3 = 2y \qquad \text{so that} \qquad y = 0, \sqrt{2}, -\sqrt{2}.$$

$y = 0$ at the end and y is never negative and so we only cross the x-axis once in between, when $y = \sqrt{2}$. In this case the real part is equal to

$$\begin{aligned} -(\sqrt{2})^6 + 9(\sqrt{2})^4 + 4 &= -8 + 32 + 4 \\ &= 36 > 0. \end{aligned}$$

The only possibility is that we go from the third quadrant to the fourth quadrant up to $y = \sqrt{2}$. The change in the argument is roughly π. From there we go into the first quadrant. Since we start and end on the real axis, the change in the argument is 0.

So the total change in the argument is approximately 4π all the way along γ. As the change in the argument is a multiple of 2π the only possibility is that the change in the argument is exactly 4π.

It follows that

$$N_0 - N_\infty = 2.$$

As $p(z)$ is holomorphic, $N_\infty = 0$ and so $p(z)$ has two zeroes in the first quadrant.

As it has three zeroes in the upper half plane and no zeroes on the imaginary axis, it follows that it has one zero in the second quadrant.