## 10. A review of contour integration

We review the definite integrals we can calculate using contour integration and the contour we use in each case.

**Example 10.1.** *Compute*

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\,\mathrm{d}x.$$

$p(x)$ and $q(x)$ are polynomials. The degree of $q(x)$ exceeds the degree of $p(x)$ by at least two and $q(x)$ has no zeroes along the real line, to ensure the indefinite integral converges. We integrate this around the standard contour, along the real axis from $[-R, R]$ and along the semicircle of radius $R$ in the upper half plane.

If $p(x)/q(x)$ is even then we can also compute

$$\int_{0}^{\infty} \frac{p(x)}{q(x)}\,\mathrm{d}x = \frac{1}{2}\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\,\mathrm{d}x.$$

**Example 10.2.** *Compute*

$$\int_{0}^{2\pi} \frac{p(\theta)}{q(\theta)}\,\mathrm{d}\theta.$$

$p(\theta)$ and $q(\theta)$ are polynomials in $\cos\theta$ and $\sin\theta$, and more generally $\cos m\theta$ and $\sin m\theta$, for any integer $m$. We assume that the denominator is nowhere zero. We integrate around the unit circle, using the substitution

$$z = e^{i\theta} \qquad \text{so that} \qquad \mathrm{d}\theta = \frac{\mathrm{d}z}{iz}$$

and the identities

$$\cos\theta = \frac{z + 1/z}{2} \qquad \text{and} \qquad \sin\theta = \frac{z - 1/z}{2i}.$$

If there is enough symmetry then we can also compute

$$\int_{0}^{\pi} \frac{p(\theta)}{q(\theta)}\,\mathrm{d}\theta.$$

**Example 10.3.** *Compute*

$$\int_{-\infty}^{\infty} \frac{p(x)\sin x}{q(x)}\,\mathrm{d}x.$$

$p(x)$ and $q(x)$ are polynomials. The degree of $q(x)$ exceeds the degree of $p(x)$ by at least two and $q(x)$ has no zeroes along the real line, to ensure the indefinite integral converges. We integrate

$$\frac{e^{iz}p(z)}{q(z)}$$

around the standard contour. We recover the original integral as the imaginary part.

As a boundary case we can also deal with the case that the degree of $q(x)$ is one more than the degree of $p(x)$, provided $p(x)/q(x)$ is odd, so that

$$\frac{p(x)\sin x}{q(x)}$$

is even. In this case we use the contour integral to comptue the Cauchy principal value and argue that this implies the indefinite integral converges.

As a further boundary case we can allow $q(x)$ to have a simple zero at zero. In this case we use an indented contour and the contour integral computes the Cauchy principal value.

**Example 10.4.** *Compute*

$$\int_0^\infty \frac{p(x)x^a}{q(x)}\,\mathrm{d}x.$$

$p(x)$ and $q(x)$ are polynomials. The degree of $q(x)$ exceeds the degree of $p(x)$ by at least two and $q(x)$ has no zeroes along the real line, to ensure the indefinite integral converges.

We have to take care of the ambiguity in $z^a$ by careful choosing a branch of the logarithm. There are two possible contours to use, an indented contour or the keyhole contour.

For the indented contour, the branch cut is along the negative imaginary axis. The indented contour only works if it is enough to compute

$$\int_{-\infty}^\infty \frac{p(x)x^a}{q(x)}\,\mathrm{d}x.$$

For the keyhole contour the branch cut is along the positive real axis. We exploit the ambiguity in $z^a$ to ensure that the integrals along $\gamma_+$ and $\gamma_-$ cancel to yield a multiple of the original integral.

**Example 10.5.** *Compute*

$$\int_0^\infty \frac{p(x)}{q(x)}\,\mathrm{d}x.$$

$p(x)$ and $q(x)$ are polynomials. The degree of $q(x)$ exceeds the degree of $p(x)$ by at least two and $q(x)$ has no zeroes along the real line, to ensure the indefinite integral converges.

We integrate

$$f(z) = \frac{p(z)\log z}{q(z)}.$$

around the keyhole contour. We exploit the ambiguity in $\log z$ to ensure that the integrals along $\gamma_+$ and $\gamma_-$ cancel to yield a multiple of the original integral.