1. Review

We start by reviewing the contents of 120A. We are interested in studying functions of one complex variable. The first big difference in comparison with real variable is that the geometry of the domain of the function plays a much more important role.

The key object is an open disk,

$$\{ z \in \mathbb{C} \mid |z - a| < r \}.$$

This is the open disk of radius r centred at a. A set U is **open** if it is a union of open disks. A set U is a **region** if it is connected and open.

All of the basic properties are expressed in terms of open disks. A power series centred at a is an expression of the form

$$\sum a_n (z - a)^n,$$

where $a_1, a_2, \dots$ is a sequence of complex numbers. Given any power series there is a quantity, $R \in [0, \infty]$, either a non-negative real number or ∞, with the following property:

- The power series converges for any point inside the open disk of radius R centred at a.
- The power series diverges for any point ouside the closed disk of radius R centred at a.
- The power series converges uniformly away from the boundary, the circle of radius R centred at a.

Nothing can be said about what happens on the circe $|z - a| = R$ (except that there is at least one point where the power series is not holomorphic on this circle).

There is even an expression for R in terms of $a_1, a_2, \dots$. Power series are the next best thing to polynomials. You can manipulate power series like polynomials, for example you can add and multiply power series.

Example 1.1.

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \dots.$$

The radius of convergence is ∞.

Example 1.2.

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \dots.$$

The radius of convergence is 1. We can build up power series for many other functions from the standard power series.

We say that a function $f\colon U \longrightarrow \mathbb{C}$ is **analytic** on a region U, if given any point $b \in U$ we can find a power series $\sum a_n(z-a)^n$ such that b is inside the open disk where $\sum a_n(z-a)^n$ converges. It turns out that one can even centre the power series at b.

The definition of a power series might seem abstract until one starts to think about a concrete example:

Example 1.3.

$$\frac{1}{\sin z}$$

is analytic on the open set

$$\{\, z \in \mathbb{C} \mid z \neq m\pi \,\}$$

the locus where $\sin z$ is not zero. What is the power series expansion about the point $3+4i$?

We say a function $f(z)$ is differentiable at a point a if

$$\lim_{z \to a} \frac{f(z)-f(a)}{z-a}$$

exists. We use the standard notation for derivatives and all of the usual rules apply.

We say that $f(z)$ is **holomorphic** at a if there is some open disk centred at a and $f(z)$ is differentiable at every point b of a.

The property of being holomorphic involves taking two dimensional limits. If we take a limit simply by approaching horizontally or approaching along vertically and equate the two answers we get the Cauchy Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

where $f = u + iv$.

One of the simplest family of holomorphic functions are Möbius transformations:

$$z \longrightarrow \frac{az+b}{cz+d}$$

where a, b, c and d are complex numbers and $ad - bc \neq 0$. This map is holomorphic except at $-d/c$. By convention this sends $-d/c$ to ∞ and ∞ to a/c. Möbius transformations send lines and circles to lines and circles, but they tend to mix the two up. Given a pair of triples a, b and c and a', b' and c' of extended complex numbers $\mathbb{C} \cup \{\infty\}$, there is a unique Möbius transformation carrying one to the other.

The main result of the course is then:

Theorem 1.4. *Let $f\colon U \longrightarrow \mathbb{C}$ be a function on a region U.*
f is holomorphic on U if and only if it is analytic on U.

To see that analytic implies holomorphic, we just need to check that a power series is differentiable wherever it converges. We can compute the derivative of a power series term by term:

$$\frac{\mathrm{d}}{\mathrm{d}z} \sum a_n(z-a)^n = \sum n a_n (z-a)^{n-1}.$$

The reason this is true is quite subtle (and, confession, hides a conceptual mistake I made in 120A). We want to use the property of uniform continuity.

It is true that if you have uniform convergence then one can automatically integrate term by term. It is not true that one can automatically differentiate term by term; there are examples in real variable where this fails.

One can differentiate term by term if we have uniform convergence of the derivative. It is not hard to use the formula for the radius of convergence to conclude that the power series for the derivative has the same radius of convergence.

To make matters worse (or better, matters better), one can differentiate term by term in complex variable, but the reasons why this is allowed use integration.

To show the reverse direction of (1.4) we need to use line integrals. If $\gamma\colon [\alpha, \beta] \longrightarrow U$ is a differentiable path in the complex plane and $f(z)$ is continuous on U then we have

$$\left| \int_\gamma f(z)\,\mathrm{d}z \right| \leq LM,$$

where L is the length of γ and M is the maximum value of $|f(z)|$ on γ.

There are a number of results proved by Cauchy. Let U be an open set whose boundary ∂U is piecewise differentiable. Let $f(z)$ be a holomorphic function on $U \cup \partial U$. We have Cauchy's theorem:

$$\int_{\partial U} f(z)\,\mathrm{d}z = 0,$$

which follows from Green's theorem and the Cauchy-Riemann equations, Cauchy's integral formula

$$f(a) = \frac{1}{2\pi i} \int_{\partial U} \frac{f(z)}{z-a}\,\mathrm{d}z = 0,$$

which follows from Cauchy's theorem, and Cauchy's formula

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_{\partial U} \frac{f(z)}{(z-a)^{n+1}}\, \mathrm{d}z = 0,$$

which follows from expanding Cauchy's integral formula as a geometric series and integrating term by term. The same trick shows that holomorphic implies analytic and we have

$$a_n = \frac{1}{2\pi i} \int_{\partial U} \frac{f(z)}{(z-a)^{n+1}}\, \mathrm{d}z = 0.$$

In fact, we can use Cauchy's formula to justify that we can differentiate term by term (or better that if $f_n(z)$ tends uniformly to $f(z)$ then $f'_n(z)$ tends uniformly to $f'(z)$).

We can use estimates to prove some basic results. For example, Liouville's theorem says that if $f(z)$ is an entire function and it is bounded then it is constant. Just integrate around larger and larger circles to conclude that $a_n = 0$ for $n > 0$.

The final topic is Laurent series. If $f(z)$ is holomorphic on the annulus

$$U = \{\, z \in \mathbb{C} \mid s < |z-a| < r \,\}$$

then it has a Laurent series expansion

$$\begin{aligned} f(z) &= \dots + \frac{a_{-3}}{(z-a)^3} + \frac{a_{-2}}{(z-a)^2} + \frac{a_{-1}}{(z-a)} + a_0 + a_1(z-a) + a_2(z-a)^2 + a_3(z-a)^3 + \dots \\ &= \sum_{n<0} a_n(z-a)^n + \sum_{n\geq 0} a_n(z-a)^n \\ &= f_\infty(z) + f_0(z), \end{aligned}$$

where $f_0(z)$ is represented by a power series which converges for $|z-a| < r$ and $f_\infty(z)$ is represented by a series which converges for $|z-a| > s$ and vanishes at ∞.

Perhaps one of the interesting cases to apply this is when $s = 0$, so that $f(z)$ has an isolated singularity at a. Isolated singularities come in three types:

Removable: In fact $a_n = 0$ if $n < 0$ (so that $f_\infty(z) = 0$). In this case $f(z)$ extends to a holomorphic function at a. This is equivalent to saying $f(z)$ is bounded near a.

Pole of order n: $a_m = 0$ if $m < n$ and $a_n \neq 0$.

$$f(z) = \frac{a_{-n}}{(z-a)^n} + \frac{a_{-n+1}}{(z-a)^{n-1}} + \dots.$$

$f(z)$ tends to infinity as z approaches a.

Essential singularity: Infinitely many negative coefficients are non-zero. $f(z)$ gets arbitrarily close to every complex number, as z approaches a.

If $f(z)$ has an isolated singularity at a the residue of $f(z)$ at a is a_{-1}, the coefficient of $(z-a)^{-1}$. The residue theorem states that if $f(z)$ has isolated singularities $a_1, a_2, \ldots, a_n$ on a region U and is otherwise holomorphic on $U \cup \partial U$ then

$$\int_{\partial U} f(z)\,\mathrm{d}z = 2\pi i \sum_{i=1}^{n} \mathrm{Res}_{a_i} f(z).$$