HOMEWORK 7, DUE WEDNESDAY MAY 20TH, 12PM

1. Find the inverse Laplace transforms of the following functions using the method of the lecture notes and ignoring issues of convergence:
(a)
$$F(s) = \frac{2s^3}{s^4 - 4}.$$
(b)
$$F(s) = \frac{2s - 2}{(s+1)(s^2 - 2s + 5)}.$$
(c)
$$F(s) = \frac{12}{s^3 + 8}.$$
2. Show that every biholomorphic map of $\mathbb{C}$ to itself is of the form $z \longrightarrow az + b$, where $a \neq 0$.
3. Show that the biholomorphic maps of the extended complex plane $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ to itself are precisely the Möbius transformations.
4. Show that every holomorphic map
$$f \colon \Delta \longrightarrow \Delta$$
is either the identity or has at most one fixed point, where a fixed point is $a \in \Delta$ such that $f(a) = a$.
5. Find the partial fraction decomposition of
$$\frac{z^6}{(z^2+1)(z-1)^2}.$$
6. Let $h(Re^{i\theta})$ be a continuous function on the circle $|z| = R$, where $R > 0$.
Show that the function
$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2rR\cos(\phi - \theta) + r^2} h(Re^{i\phi})\,\mathrm{d}\phi$$
is harmonic on the open disk
$$U = \{\, z \in \mathbb{C} \mid |z| < R \,\}$$
and extends to a continuous function on the closed disk which restricts to $h(Re^{i\theta})$.

Challenge Problems: (Just for fun)

7. (a) Show that every biholomorphic map of $U = \mathbb{C} - \{a_1, a_2, \ldots, a_n\}$, the complex plane punctured at finitely many points, is a Möbius transformation that permutes the points of

$$\{ a_1, a_2, \ldots, a_n, \infty \}.$$

(b) Find the biholomorphic maps of $U = \mathbb{C} - \{0, 1\}$.
(c) Find the biholomorphic maps of $U = \mathbb{C} - \{-1, 0, 1\}$.
(d) Find the biholomorphic maps of $U = \mathbb{C} - \{-1, 0, 2\}$.

8. Let $f\colon \Delta \longrightarrow \Delta$ be a holomorphic map that is not biholomorphic. Show that if f has a fixed point a and f_n is the nth iterate of f (that is, compose f with itself n times) then the sequence of points

$$b \qquad f_1(b) = f(b) \qquad f_2(b) = f(f(b)) \qquad \ldots$$

converges to a, for any $b \in \Delta$.