HOMEWORK 5, DUE WEDNESDAY MAY 6TH, 12PM

1. Show the following functions are harmonic and find harmonic conjugates:
(a)

$$x^2 - y^2$$

(b)

$$xy + 3x^2y - y^3$$

(c)

$$\sinh x \sin y$$

(d)

$$\frac{x}{x^2 + y^2}.$$

2. Show that if v is harmonic conjugate for u then $-u$ is a harmonic conjugate of v.
3. (a) Show that Laplace's equation in polar coordinates is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

(b) Show that $\ln |z|$ is harmonic on the punctured plane $\mathbb{C} \setminus \{0\}$.
(c) Show that $\ln |z|$ has no conjugate harmonic function on the punctured plane $\mathbb{C}\setminus\{0\}$, but that it does on the plane minus the non-positive reals, $\mathbb{C} \setminus (-\infty, 0]$.
(d) Show that $u(re^{i\theta}) = \theta \ln r$ is harmonic. Find a harmonic conjugate v for u, using the polar form of the Cauchy-Riemann equations. What is the holomorphic function $u + iv$?
4. Let u be a harmonic function on the annulus

$$\{\, z \in \mathbb{C} \mid a < |z| < b \,\}.$$

Show that there is a constant C such that $u(z) - C \ln |z|$ has a harmonic conjugate on the annulus. Show that this constant is

$$C = \frac{r}{2\pi} \int_0^{2\pi} \frac{\partial u}{\partial r}(re^{i\theta})\, \mathrm{d}\theta,$$

where $r \in (a, b)$.
5. Let U be a bounded region and let u be a harmonic function that extends continuously to the boundary ∂U of U.
Show that if $u \in [a, b]$ on ∂U then $u \in [a, b]$ on the whole of U.

6. Fix $n \geq 1$, $r > 0$ and $\lambda = \rho e^{i\theta}$. What is the maximum modulus of $z^n + \lambda$ over the closed disk

$$\{ z \in \mathbb{C} \,|\, |z| \leq r \}?$$

Where does $z^n + \lambda$ attain its maximum over this disk?

7. Let $f(z)$ be a holomorphic function on a region U that is nowhere zero on U.

(a) Show that if $|f(z)|$ attains its minimum on U then $f(z)$ is constant.

(b) If U is bounded and $f(z)$ extends to a continuous function on the boundary of U then $|f(z)|$ attains its minimum on ∂U.

8. Let $f(z)$ be a bounded holomorphic function on the right half plane. Suppose that $f(z)$ extends continuously to the imaginary axis and that

$$|f(iy)| \leq M$$

for all points iy on the imaginary axis. Show that

$$|f(z) \leq M$$

for all z in the right half plane. (*Hint: consider* $(z+1)^{-\epsilon} f(z)$ *on a large half disk, where* $\epsilon > 0$ *is small*).

Challenge Problems: (Just for fun)

9. Prove the fundamental theorem of algebra by applying the maximum principle to $1/p(z)$ on a disk of large radius.