HOMEWORK 0, DUE TWELTH OF NEVER

1. What are the first four terms of the power series of

$$\frac{e^{2z}\sin 5z}{1-z}$$

centred at 0.
2. Show that the function

$$z \longrightarrow \frac{e^{2z}\sin(5z-1)}{1-z}$$

is holomorphic, except at 1. Conclude that this function has a power series expansion based at any point a. What is the radius of convergence?
3. Write down the unique Möbius transformation carrying 0 to i, 1 to 1 and ∞ to -1. What happens to the upper half plane $\mathbb{H}$?
4. Calculate

$$\oint_{|z-a|=r} (z-a)^m \, \mathrm{d}z$$

where m is an integer.
(i) Directly.
(ii) Using results of Cauchy.
5. Find all Laurent expansions of

$$\frac{1}{(z^2-1)(z^2-9)},$$

centred at 0.
6. What types of singularities does

$$\frac{ze^z}{z^2-1}$$

have?
7. Calculate

$$\operatorname{Res}_0 \frac{\sin z}{z^2}.$$

Challenge Problems: (Just for fun)

8. Let

$$M\colon \mathbb{P}^1 \longrightarrow \mathbb{P}^1 \qquad \text{given by} \qquad M(z) = \frac{az+b}{cz+d}$$

be a Möbius transformation. A fixed point p of M is a point of $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, a complex number or ∞, such that

$$M(p) = p.$$

Show that the number of fixed points is one of

(1) 1;
(2) 2;
(3) Every point $p \in \mathbb{P}^1$.

Give examples in all three cases.

9. Let U be a region in the plane. Let f be a continuous function on U such that

$$\int_\gamma f(z)\,\mathrm{d}z = 0,$$

for every closed path in U.

Morera's theorem: Show that $f(z)$ is holomorphic. (*Hint: try to define a function by the 'rule':*

$$F(z) = \int_{z_0}^{z} f(z)\,\mathrm{d}z.$$

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10. Let $f_1, f_2, \dots$ be a sequence of holomorphic functions which tends uniformly to a function f on a region U.

Show that f is holomorphic and that $f_1', f_2', \dots$ tends uniformly to f'.