

**TAKE HOME FINAL EXAM
MATH 120B, UCSD, SPRING 20**

You have 24 hours.

There are 6 problems, and the total number of points is 100.

Please make your work as clear and easy to follow as possible. There is no need to be verbose but explain all of the steps, using your own words. You may consult the lecture notes and model answers but you may not use any other reference nor may you confer with anyone. You may use any of the standard results in the lecture notes as long as you clearly state what you are using. If you don't know how to solve the whole problem answer the portion you can solve.

Please submit your answers on Gradescope by 1pm on Thursday June 11th.

1. (10pts) Show that $u(x, y) = e^{-3x} \cos 3y$ is harmonic and find a harmonic conjugate.
2. (10pts) Find a biholomorphic map between the unit disk Δ and the region

$$U = \left\{ z \in \mathbb{C} \mid 0 < \text{Arg}(z) < \frac{\pi}{4} \right\}.$$

3. (20pts) Let $a \in \mathbb{C}$ be a constant.
 - (a) Find the inverse Laplace transform of

$$F(s) = \frac{s - 1}{s^3 + 2s^2 + s}$$

using the Bromwich integral.

- (b) Find the inverse Laplace transform of

$$F(s) = \frac{1}{s + a}$$

using the Bromwich integral.

- (c) Find the inverse Laplace transform of

$$F(s) = \frac{1}{(s + a)^2}$$

using the Bromwich integral.

- (d) Decompose the rational function

$$\frac{s - 1}{s^3 + 2s^2 + s}$$

into principal parts and use this together with (b) and (c) to find the inverse Laplace transform.

4. (20pts) Show that

$$\int_{-\infty}^{\infty} \frac{x \sin \alpha x}{x^2 + \beta^2} dx = \pi \operatorname{sgn} \alpha e^{-|\alpha\beta|}$$

where $\alpha \neq 0$, $\beta \neq 0$ and

$$\operatorname{sgn} \alpha = \begin{cases} 1 & \text{if } \alpha > 0 \\ -1 & \text{if } \alpha < 0. \end{cases}$$

(Hint: First do the case $\alpha > 0$ and $\beta > 0$.)

5. (20pts) Compute

$$\int_0^{\infty} \frac{x^a}{x^2 + x + 1} dx \quad \text{where} \quad a \in (-1, 1).$$

6. (20pts) Show that the equation $z^4 + z + 1 = 0$ has one solution in each quadrant. Prove that all solutions lie inside the open disk

$$\left\{ z \in \mathbb{C} \mid |z| < \frac{3}{2} \right\}.$$

7. (Extra credit: 10pts) If $f(z)$ is a holomorphic function on the first quadrant that approaches a real number as the imaginary part of z goes to zero and that approaches an imaginary number as the real part of z goes to zero then show that $f(z)$ extends to an entire function that is odd (that is, $f(-z) = -f(z)$).

8. (Extra credit: 20pts) (a) Show that every biholomorphic map f of $U = \mathbb{C} - \{a_1, a_2, \dots, a_n\}$, the complex plane punctured at finitely many points, is a Möbius transformation that permutes the points

$$a_1, a_2, \dots, a_n \quad \text{and} \quad \infty.$$

(you may assume that if f is a rational function then it is a Möbius transformation).

(b) Find the biholomorphic maps of $U = \mathbb{C} - \{0, 1\}$.