TAKE HOME FINAL EXAM
MATH 120B, UCSD, SPRING 20

You have 24 hours.

There are 7 problems, and the total number of points is 110.

Please make your work as clear and easy to follow as possible. There is no need to be verbose but explain all of the steps, using your own words. You may consult the lecture notes and model answers but you may not use any other reference nor may you confer with anyone. You may use any of the standard results in the lecture notes as long as you clearly state what you are using. If you don't know how to solve the whole problem answer the portion you can solve.

Please submit your answers on Gradescope by 1pm on Thursday June 11th.

1. (10pts) Show that $u(x, y) = e^{-3x} \cos 3y$ is harmonic and find a harmonic conjugate.

There are two ways to approach this question. The first is just to guess a holomorphic function whose real part is u. One obvious thing to try (with perhaps a little bit of trial and error) is

$$f(z) = e^{-3z}.$$

In this case

$$\begin{aligned} e^{-3z} &= e^{-3(x+iy)} \\ &= e^{-3x}e^{-3iy} \\ &= e^{-3x}(\cos 3y - i \sin 3y) \\ &= e^{-3x} \cos 3y - ie^{-3x} \sin 3y. \end{aligned}$$

It follows that the real part is

$$e^{-3x} \cos 3y = u(x, y).$$

Thus u is harmonic, as it is the real part of a holomorphic function. A harmonic conjugate is given by the imaginary part, which is

$$v(x, y) = -e^{-3x} \sin 3y.$$

We can also check directly that u is harmonic. We have

$$\frac{\partial u}{\partial x} = -3e^{-3x} \cos 3y \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -3e^{-3x} \sin 3y.$$

Therefore

$$\begin{aligned} \Delta u &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \\ &= 9e^{-3x} \cos 3y - 9e^{-3x} \cos 3y \\ &= 0. \end{aligned}$$

Thus u is harmonic.

To find a harmonic conjugate, we can solve the Cauchy-Riemann equations:

$$\frac{\partial v}{\partial y} = -3e^{-3x} \cos 3y \qquad \text{and} \qquad \frac{\partial v}{\partial x} = 3e^{-3x} \sin 3y.$$

If we integrate the first equation with respect to y then we get

$$\begin{aligned} v(x, y) &= \int -3e^{-3x} \cos 3y \, dy \\ &= -e^{-3x} \sin 3y + h(x). \end{aligned}$$

Here $h(x)$ is a constant of integration that depends on x. If we plug this into the second equation we get

$$h'(x) - 3e^{-3x}\sin 3y = -3e^{-3x}\sin 3y.$$

Thus we can take $h(x) = 0$ and $v(x, y) = -e^{-3x}\sin 3y$, the same answer as before.

2. (10pts) Let

$$f\colon U \longrightarrow \mathbb{C} \qquad \text{given by} \qquad f(z) = z^4.$$

As $z \longrightarrow z^4$ is an entire function it follows that f is holomorphic. As

$$\mathbb{H} = \{\, z \in \mathbb{C} \,|\, 0 < \operatorname{Arg}(z) < \pi \,\}$$

and the map $z \longrightarrow z^4$ multiplies the argument by 4 it follows that the image of U is $\mathbb{H}$. We can define a holomorphic branch of $z \longrightarrow z^{1/4}$ on $\mathbb{C} - (-\infty, 0]$, by using the principal value of the logarithm on V,

$$\operatorname{Log} z = \ln|z| + i\operatorname{Arg}(z) \qquad \text{where} \qquad \operatorname{Arg}(z) \in (-\pi, \pi).$$

Thus f defines a biholomorphic map between U and $\mathbb{H}$.

Now we are looking for a biholomorphic map of the upper half plane to the unit disk. The Möbius transformation

$$M(z) = \frac{z - i}{z + i}$$

sends 0 to -1, ∞ to 1 and 1 to

$$\begin{aligned} \frac{1-i}{1+i} &= \frac{(1-i)(1-i)}{2} \\ &= -i. \end{aligned}$$

Note that -1, 1 and $-i$ are three points of the unit circle. As Möbius transformations send lines and circles to lines and circles, and three points determine a circle it follows that M sends the real axis to the unit circle.

As i is sent to zero, it follows that the upper half plane is sent to the unit disk. Hence the map

$$z \longrightarrow \frac{z^4 - i}{z^4 + i}$$

defines a biholomorphic map between U and Δ.

3. (20pts) For the Bromwich integral, we have to compute the sum of the residues of $e^{st}F(s)$.

(a)

$$\frac{s-1}{s^3 + 2s^2 + s}$$

has isolated singularities at the zeroes of

$$s^3 + 2s^2 + s = s(s+1)^2.$$

Thus $e^{st}F(s)$ has a simple pole at 0 and a double pole at -1. We have

$$\begin{aligned}\operatorname{Res}_0 e^{st}F(s) &= \lim_{s\to 0} \frac{(s-1)e^{st}}{(s+1)^2} \\ &= -1.\end{aligned}$$

We also have

$$\begin{aligned}\operatorname{Res}_{-1} e^{st}F(s) &= \lim_{s\to -1} \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{(s-1)e^{st}}{s}\right) \\ &= \lim_{s\to -1} \frac{\mathrm{d}}{\mathrm{d}s}\left(e^{st} - \frac{e^{st}}{s}\right) \\ &= \lim_{s\to -1} \left(te^{st} - \frac{tse^{st} - e^{st}}{s^2}\right) \\ &= te^{-t} + te^{-t} + e^{-t} \\ &= (1+2t)e^{-t}.\end{aligned}$$

The Laplace transform is therefore:

$$\begin{aligned}f(t) &= \operatorname{Res}_0 e^{st}F(s) + \operatorname{Res}_{-1} e^{st}F(s) \\ &= -1 + (1+2t)e^{-t}.\end{aligned}$$

(b)

$$\frac{1}{s+a}$$

has a simple pole at $s = -a$. We have

$$\begin{aligned}\operatorname{Res}_{-a} e^{st}F(s) &= \lim_{s\to -a} e^{st} \\ &= e^{-at}.\end{aligned}$$

(c)

$$\frac{1}{(s+a)^2}$$

has a double pole at $s = -a$. We have

$$\begin{aligned}\operatorname{Res}_{-a} e^{st}F(s) &= \lim_{s\to -a} \frac{\mathrm{d}}{\mathrm{d}s} e^{st} \\ &= \lim_{s\to -a} te^{st} \\ &= te^{-at}.\end{aligned}$$

(d)

$$F(s) = \frac{s-1}{s^3+2s^2+s}$$

has a simple pole at $s=0$ and a double pole $s=-1$. For the principal part at $s=0$ and $s=-1$ we have

$$P_0(s) = \frac{\alpha}{s} \qquad \text{and} \qquad P_{-1}(s) = \frac{\beta}{(s+1)^2} + \frac{\gamma}{s+1}.$$

We have

$$\begin{aligned} \alpha &= \operatorname{Res}_0 F(s) \\ &= \lim_{s\to 0} \frac{s-1}{(s+1)^2} \\ &= -1, \end{aligned}$$

we also have

$$\begin{aligned} \beta &= \operatorname{Res}_{-1} (s+1)F(s) \\ &= \operatorname{Res}_{-1} \frac{s-1}{s(s+1)} \\ &= \lim_{s\to -1} \frac{s-1}{s} \\ &= 2, \end{aligned}$$

and finally

$$\begin{aligned} \gamma &= \operatorname{Res}_{-1} F(s) \\ &= \lim_{s\to -1} \frac{\mathrm{d}}{\mathrm{d}s} \frac{s-1}{s} \\ &= \lim_{s\to -1} \frac{1}{s^2} \\ &= 1. \end{aligned}$$

It follows that

$$\begin{aligned} F(s) &= P_0(s) + P_{-1}(s) \\ &= -\frac{1}{s} + \frac{2}{(s+1)^2} + \frac{1}{s+1}. \end{aligned}$$

Linearity of the Laplace transform and (b) and (c) imply that the inverse Laplace transform is

$$f(t) = -1 + 2te^{-t} + e^{-t},$$

the same as in part (a).

4. (20pts) We first suppose that $\alpha > 0$ and $\beta > 0$. We integrate

$$f(z) = \frac{ze^{i\alpha z}}{z^2 + \beta^2}$$

over the standard contour $\gamma = \gamma_1 + \gamma_2$, where γ_1 describes the interval from $-R$ to R along the real axis and γ_2 is the semicircle of radius R in the upper half plane starting at R and ending at $-R$. $f(z)$ has poles at $\pm i\beta$. Only the pole at $i\beta$ is in the upper half plane. We assume that $R > \beta$ so that we capture this isolated singularity.
As we have a simple pole, the residue at $i\beta$ is:

$$\begin{aligned}
\operatorname{Res}_{i\beta} f(z) &= \lim_{z \to i\beta} \frac{z(z - i\beta)e^{i\alpha z}}{z^2 + \beta^2} \\
&= \lim_{z \to i\beta} \frac{ze^{i\alpha z}}{z + i\beta} \\
&= \frac{i\beta e^{-\alpha\beta}}{i\beta + i\beta} \\
&= \frac{e^{-\alpha\beta}}{2}.
\end{aligned}$$

The residue theorem implies that

$$\begin{aligned}
\int_\gamma \frac{ze^{i\alpha z}}{z^2 + \beta^2}\, \mathrm{d}z &= 2\pi i \operatorname{Res}_{i\beta} f(z) \\
&= \pi i e^{-\alpha\beta} \\
&= \pi i \operatorname{sgn} \alpha e^{-|\alpha\beta|}.
\end{aligned}$$

Next we show the integral over the semicircle goes to zero as we increase R to infinity. We have

$$\begin{aligned}
\left| \int_{\gamma_2} \frac{ze^{i\alpha z}}{z^2 + \beta^2}\, \mathrm{d}z \right| &\leq \int_{\gamma_2} \frac{|ze^{i\alpha z}|}{|z^2 + \beta^2|} |\mathrm{d}z| \\
&\leq \frac{R}{R^2 - \beta^2} \int_{\gamma_2} |e^{i\alpha z}\, |\mathrm{d}z| \\
&< \frac{\pi R}{\alpha(R^2 - \beta^2)},
\end{aligned}$$

which goes to zero, as R goes to infinity. To get from line two to line three we applied Jordan's Lemma, see Question 0 of Homework 2.
Taking the limit as R approaches ∞ we get

$$\lim_{R \to \infty} \int_{-R}^{R} \frac{xe^{i\alpha x}}{x^2 + \beta^2}\, \mathrm{d}x = \pi i \operatorname{sgn} \alpha e^{-|\alpha\beta|}.$$

Taking the imaginary part of both sides we get

$$\lim_{R\to\infty}\int_{-R}^{R}\frac{x\sin\alpha x}{x^2+\beta^2}\,\mathrm{d}x=\pi\operatorname{sgn}\alpha e^{-|\alpha\beta|}.$$

As the integrand

$$\frac{x\sin\alpha x}{x^2+\beta^2}$$

is even, convergence of the Cauchy principal value implies convergence of the improper integral to the same limit. Thus

$$\int_{-\infty}^{\infty}\frac{x\sin\alpha x}{x^2+\beta^2}\,\mathrm{d}x=\pi\operatorname{sgn}\alpha e^{-|\alpha\beta|}.$$

Now we consider what happens if we allow $\beta<0$. The LHS doesn't change, as $(-\beta)^2=\beta^2$. The RHS is also unchanged as $|\alpha\cdot-\beta|=|\alpha\beta|$. Finally we consider what happens if we allow $\alpha<0$. The LHS changes sign. The RHS also changes sign, as $\operatorname{sgn}(-\alpha)=-\operatorname{sgn}\alpha$ and $|-\alpha\cdot\beta|=|\alpha\beta|$. Thus the formula is correct, no matter the sign of α and β.
5. (20pts) Let

$$f(z)=\frac{z^a}{z^2+z+1}.$$

To make sense of z^a we need to choose a branch of the logarithm. We cut the complex plane along the non-negative real axis:

$$V=\mathbb{C}\setminus[0,\infty).$$

We then put

$$\log z=\ln|z|+i\arg z\qquad\text{where}\qquad\arg z\in[0,2\pi).$$

This defines a holomorphic branch of the logarithm on V. Armed with the logarithm it is then straightforward to define

$$z^a=e^{a\log z}$$

and this makes $f(z)$ a holomorphic function on V away from the zeroes of z^2+z+1.
We integrate $f(z)$ over the keyhole contour,

$$\gamma=\gamma_s+\gamma_-+\gamma_++\gamma_b,$$

where γ_s goes clockwise around the circle of radius $\rho>0$ centred at the origin, starting and ending at ρ, γ_+ goes from ρ to R just above the cut, γ_b goes anticlockwise around the circle of radius $R>0$ centred at the origin, starting and ending at R and γ_- goes from R to ρ just below the cut.
Note that

$$(z^2+z+1)(z-1)=z^3-1.$$

Thus $f(z)$ has isolated singularities at the two cube roots of unity:

$$\omega = e^{2\pi i/3} \qquad \text{and} \qquad \omega^2 = e^{4\pi i/3}.$$

If $\rho < 1$ and $R > 1$ then we capture both poles. As the poles are simple we have

$$\begin{aligned} \operatorname{Res}_\omega f(z) &= \lim_{z\to\omega} \frac{z^a}{z-\omega^2} \\ &= \frac{\omega^a}{\omega-\omega^2} \\ &= -i\frac{\omega^a}{\sqrt{3}} \end{aligned}$$

and

$$\begin{aligned} \operatorname{Res}_{\omega^2} f(z) &= \lim_{z\to\omega^2} \frac{z^a}{z-\omega} \\ &= \frac{\omega^{2a}}{\omega^2-\omega} \\ &= i\frac{\omega^{2a}}{\sqrt{3}} \end{aligned}$$

The residue theorem implies that

$$\begin{aligned} \int_\gamma f(z) = \frac{z^a}{z^2+z+1}\,\mathrm{d}z &= 2\pi i\left(\operatorname{Res}_\omega f(z) + \operatorname{Res}_{\omega^2} f(z)\right) \\ &= -\frac{2\pi}{\sqrt{3}}(\omega^{2a}-\omega^a) \\ &= \frac{2\pi}{\sqrt{3}}(e^{2\pi ia/3}-e^{4\pi ia/3}). \end{aligned}$$

Next we show the integrals over the two circles go to zero as we increase R to infinity and we decrease ρ to zero. For the big circle the value of $|f(z)|$ is at most

$$\begin{aligned} |f(z)| &\leq \left|\frac{z^a}{z^2+z+1}\right| \\ &\leq \frac{|z^a|}{|z^2+z+1|} \\ &\leq \frac{R^a}{R^2-R-1}. \end{aligned}$$

Therefore we have

$$\begin{aligned}\left|\int_{\gamma_b} \frac{z^a}{z^2+z+1}\,\mathrm{d}z\right| &\le LM \\ &\le \frac{2\pi R^{a+1}}{R^2-R-1}\end{aligned}$$

which goes to zero, as R goes to infinity, as $a+1<2$.
For the small circle the value of $|f(z)|$ is at most

$$\begin{aligned}|f(z)| &\le \left|\frac{z^a}{z^2+z+1}\right| \\ &\le \frac{|z^a|}{|z^2+z+1|} \\ &\le \frac{\rho^a}{1-\rho-\rho^2}.\end{aligned}$$

Therefore we have

$$\begin{aligned}\left|\int_{\gamma_s} \frac{z^a}{z^2+z+1}\,\mathrm{d}z\right| &\le LM \\ &\le \frac{\rho^{a+1}}{1-\rho-\rho^2}.\end{aligned}$$

which goes to zero, as ρ goes to zero, as $a+1>0$.
For the remaining integrals, the branch of the logarithm becomes important. For the integral above the cut we have

$$\log z = \ln x \qquad \text{so that} \qquad z^a = x^a.$$

For the integrals below the cut we have

$$\log z = \ln x + 2\pi i \qquad \text{so that} \qquad z^a = x^a e^{2a\pi i}.$$

For the integral over γ_+ we have

$$\int_{\gamma_+} \frac{z^a}{z^2+z+1}\,\mathrm{d}z = \int_\rho^R \frac{x^a}{x^2+x+1}\,\mathrm{d}x.$$

By contrast, for the integral over γ_- we have

$$\int_{\gamma_-} \frac{z^a}{z^2+z+1}\,\mathrm{d}z = -e^{2a\pi i}\int_\rho^R \frac{x^a}{x^2+x+1}\,\mathrm{d}x.$$

Let I be the Cauchy principal value of

$$\int_\rho^R \frac{x^a}{x^2+x+1}\,\mathrm{d}x.$$

If we let ρ go to zero and R go to infinity then the integral over γ_+ approaches I and the integral over γ_- approaches $-e^{2a\pi i}I$.

It follows that

$$(1 - e^{2a\pi i})I = 2\pi(e^{2\pi i a/3} - e^{4\pi i a/3}).$$

Solving for I gives:

$$\begin{aligned} I &= \frac{2\pi}{\sqrt{3}} \frac{e^{2\pi i a/3} - e^{4\pi i a/3}}{1 - e^{2a\pi i}} \\ &= \frac{2\pi}{\sqrt{3}} \frac{e^{-\pi i a/3} - e^{\pi i a/3}}{e^{-a\pi i} - e^{a\pi i}} \\ &= \frac{2\pi}{\sqrt{3}} \frac{\sin \pi a/3}{\sin \pi a}. \end{aligned}$$

6. (20pts) We first locate the real zeroes. Consider

$$x^4 - x + 1.$$

If $x < 0$ then all three terms are positive so that $x^4 - x + 1 > 0$. If $x \in [0, 1)$ then $1 - x > 0$ and $x^4 \geq 0$ so that $x^4 - x + 1 > 0$. If $x \geq 1$ then $x^4 - x \geq 0$ so that $x^4 - x + 1 > 0$. Thus $z^4 - z + 1$ is always positive on the real axis.

Consider the change in the argument as we go around the closed contour

$$\gamma = \gamma_1 + \gamma_2 + \gamma_3,$$

where γ_1 traverses the interval $[0, R]$, γ_2 traverses the quarter circle in the first quadrant of radius R centred at the origin, starting at R and ending at iR and γ_3 traverses the imaginary axis, starting at iR and ending at the origin.

There is no chanage in the argument along γ_1 as $z^4 - z + 1$ is real and positive along the real axis. If R is big enough then the dominant term is z^4 and so the change in the argument is

$$4 \cdot \frac{\pi}{2} = 2\pi.$$

Now we consider what happens along γ_3. If $z = iy$ then

$$z^4 - z + 1 = (y^4 + 1) - iy.$$

If $y = R$ then the dominant term is the real part which is large and positive. The imaginary part is then negative, but small in comparison to the real part. So we start in the fourth quadrant.

We now determine how often we cross the real axis. This is when $y = 0$. Thus we never cross the real axis. It follows that the change in the argument over γ_3 is close to zero.

In total as go around γ the change in the argument is 2π. Thus $z^4 - z + 1$ has one zero in the first quadrant. Since the coefficients of $z^4 - z + 1$

are reals, it must have the same number of zeroes above and below the real axis.
As there are four zeroes in total, the only possibility is that there is one zero in each quadrant.
To find the number of solutions in the open disk of radius 3/2 we apply Rouche's theorem to the circle of radius 3/2 centred at zero. We try $f(z) = z^4$ and $h(z) = 1 - z$. On the circle $|z| = 3/2$ we have

$$\begin{aligned} |h(z)| &= |1 - z| \\ &\leq 1 + |z| \\ &= 1 + \frac{3}{2} \\ &= \frac{5}{2} \\ &< \frac{81}{16} \\ &= |z|^4 \\ &= |f(z)|. \end{aligned}$$

Thus Rouche's theorem implies that

$$f(z) + h(z) = z^4 - z + 1$$

has the same number of zeroes as $f(z) = z^4$ on the open disk $|z| < 3/2$. But z^4 has a zero of order four at 0 and so all four zeroes of $z^4 - z + 1$ lie in the circle of radius 3/2 centred at 0.
7. (10pts) As the imaginary part of $f(z)$ goes to zero as we approach the real axis it follows by the refelction principle that $f(z)$ extends to a holomorphic function on the right half plane

$$\{ z \in \mathbb{C} \mid \operatorname{Re}(z) > 0 \}$$

and $f(z)$ satisfies the equation

$$f(\bar{z}) = \overline{f(z)}.$$

Now consider what happens as we approach the imaginary axis. By assumption the real part goes to zero. Thus we can extend $f(z)$ to a holomorphic function on the whole complex plane and

$$f(z^*) = f(z)^*,$$

where * represents reflection across the imaginary axis. The reflection of $z = x + iy$ is $-x + iy$.

Suppose that $f(z) = u + iv$. We compute

$$\begin{aligned} f(-z) &= f(-x - iy) \\ &= -u(x - iy) + iv(x - iy) \\ &= -u(x + iy) - iv(x + iy) \\ &= -f(z). \end{aligned}$$

To get from the first line to the second line we used reflection across the imaginary axis and to get from the second line to the third line we used reflection across the real axis.

It follows that $f(z)$ extends to an entire function that is odd.

8. (a) If $f: U \longrightarrow U$ is biholomorphic then f has finitely many isolated singularities on $\mathbb{C}$. Consider the behaviour near an isolated singularity. Either f has a removable singularity, a pole or an essential singularity. f cannot have an essential singularity, since then by Casorati-Weierstrass f approaches arbitrarily close to every complex number. This is not possible as f is a bijection.

Thus f has only poles as singularities. It follows that f is a meromorphic function. But then f is a rational function, the quotient of two polynomials:

$$f(z) = \frac{p(z)}{q(z)}.$$

Thus f is a Möbius transformation.

(b) We are looking for Möbius transformations that permute the three points 0, 1 and ∞. Suppose M and N induce the same permutation of 0, 1 and ∞. If P is the inverse of N then the composition $M \circ P$ fixes all three points. But then $M \circ P$ is the identity. It follows that $M = N$.

There are only six ways to permute these three points, so we are looking for at most six different Möbius transformations.

$$z \longrightarrow z$$

is the identity and it fixes all three of 0, 1 and ∞.

$$z \longrightarrow 1 - z$$

sends 0 to 1, 1 to 0 and fixes ∞.

$$z \longrightarrow \frac{1}{z}$$

sends 0 to ∞, ∞ to 0 and fixes 1.

$$z \longrightarrow \frac{z}{z-1}$$

sends 1 to ∞, ∞ to 1 and 0 to 0.

$$z \longrightarrow \frac{z-1}{z}$$

sends 0, ∞, ∞ to 1 and 1 to 0. Finally

$$z \longrightarrow \frac{1}{1-z}$$

sends 0 to 1, 1 to ∞ and ∞ to 0.