MODEL ANSWERS TO THE NINTH HOMEWORK

5.1.2. (a) Suppose that we write

$$x^2 = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}.$$

We just need to compute

$$\begin{aligned}
A_m &= 2\int_0^1 \phi(x) \sin m\pi x \, \mathrm{d}x \\
&= 2\int_0^1 x^2 \sin m\pi x \, \mathrm{d}x \\
&= 2\left[\frac{-x^2}{m\pi} \cos m\pi x + \frac{2x}{m^2\pi^2} \sin m\pi x + \frac{2}{m^3\pi^3} \cos m\pi x\right]_0^1 \\
&= -\frac{2}{m\pi} \cos m\pi + \frac{4}{m^3\pi^3} \cos m\pi - \frac{4}{m^3\pi^3} \\
&= \frac{2}{m\pi}(-1)^{m+1} + \frac{4}{m^3\pi^3}(-1)^m - \frac{4}{m^3\pi^3}
\end{aligned}$$

It follows that

$$A_m = \begin{cases} -\frac{2}{m\pi} & \text{if } m \text{ is even} \\ \frac{2}{m\pi} - \frac{8}{m^3\pi^3} & \text{if } m \text{ is odd.} \end{cases}$$

Thus

$$x^2 = \frac{2}{\pi}\left(\sin \pi x - \frac{1}{2}\sin 2\pi x + \frac{1}{3}\sin 3\pi x + \dots\right) - \frac{8}{\pi^3}\left(\sin \pi x + \frac{1}{27}\sin 3\pi x + \frac{1}{125}\sin 5\pi x + \dots\right).$$

(b) Suppose that we write

$$x^2 = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{l}.$$

We just need to compute

$$\begin{aligned}
A_m &= 2\int_0^1 \phi(x)\cos m\pi x\,\mathrm{d}x \\
&= 2\int_0^1 x^2\cos m\pi x\,\mathrm{d}x \\
&= 2\left[\frac{x^2}{m\pi}\sin m\pi x + \frac{2x}{m^2\pi^2}\cos m\pi x - \frac{2}{m^3\pi^3}\sin m\pi x\right]_0^1 \\
&= \frac{4}{m^2\pi^2}\cos m\pi \\
&= \frac{4}{m^2\pi^2}(-1)^m.
\end{aligned}$$

On the other hand

$$\begin{aligned}
A_0 &= 2\int_0^1 \phi(x)\,\mathrm{d}x \\
&= 2\int_0^1 x^2\,\mathrm{d}x \\
&= 2\left[\frac{x^3}{3}\right]_0^1 \\
&= \frac{2}{3}.
\end{aligned}$$

Thus

$$x^2 = \frac{1}{3} + \frac{4}{\pi^2}\left(-\sin\pi x + \frac{1}{4}\sin 2\pi x - \frac{1}{9}\sin 3\pi x + \dots\right).$$

5.1.5. (a) We start with

$$x = \frac{2l}{\pi}\left(\sin\frac{\pi x}{l} - \frac{1}{2}\sin\frac{2\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + \dots\right).$$

If we integrate both sides and assume that we can switch the order of integration and summation we get

$$\frac{x^2}{2} = c - \frac{2l^2}{\pi^2}\left(\cos\frac{\pi x}{l} - \frac{1}{4}\cos\frac{2\pi x}{l} + \frac{1}{9}\cos\frac{3\pi x}{l} + \dots\right),$$

where c is a constant to be determined.
If we integrate both sides over the interval $(0, l)$ then every term but the first is zero. Thus

$$\frac{l^3}{6} = cl,$$

so that

$$c = \frac{l^2}{6}$$

and so

$$\frac{x^2}{2} = \frac{l^2}{6} - \frac{2l^2}{\pi^2}\left(\cos\frac{\pi x}{l} - \frac{1}{4}\cos\frac{2\pi x}{l} + \frac{1}{9}\cos\frac{3\pi x}{l} + \dots\right),$$

It is reassuring that this answer is consistent with the answer in 1(b).
(b) If we set $x = 0$ the LHS is zero. Thus

$$0 = \frac{l^2}{6} - \frac{2l^2}{\pi^2}\left(1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{25} + \dots\right).$$

It follows that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \frac{\pi^2}{12}.$$

5.1.8. Following the hint, we first find the equilibrium solution $U(x)$. By definition the equilibrium solution is independent of time. This the heat equation reduces to

$$U_{xx} = 0 \qquad \text{subject to} \qquad U(0) = 0, U(1) = 1.$$

The general solution of the PDE is

$$U(x) = Ax + B.$$

The boundary conditions imply that

$$B = 0 \qquad \text{and} \qquad A = 1.$$

Thus $U(x) = x$ is the equilibrium solution.
Now consider the diffusion equation with initial conditions

$$u(x,0) = \phi(x) - x.$$

Thus

$$u(x,0) = \begin{cases} \frac{3x}{2} & \text{for } 0 < x < \frac{2}{3} \\ 3 - 3x & \text{for } \frac{2}{3} < x < 1. \end{cases}$$

Now the boundary conditions are the Dirichlet conditions $u(0,t) = u(1,t) = 0$. Thus we expand $u(x,0)$ as a Fourier sine series. The coefficients are

$$
\begin{aligned}
A_m &= 2\int_0^1 \phi(x)\sin m\pi x\,\mathrm{d}x \\
&= \int_0^{2/3} 3x\sin m\pi x\,\mathrm{d}x + 6\int_{2/3}^1 (1-x)\sin m\pi x\,\mathrm{d}x \\
&= \left[\frac{3x}{m\pi}\cos m\pi x + \frac{3}{m^2\pi^2}\sin m\pi x\right]_0^{2/3} + 6\left[\frac{1-x}{m\pi}\cos m\pi x - \frac{1}{m^2\pi^2}\sin m\pi x\right]_{2/3}^1 \\
&= \frac{2}{m\pi}\cos\frac{2m\pi}{3} + \frac{3}{m^2\pi^2}\sin\frac{2m\pi}{3} - \frac{2}{m\pi}\cos\frac{2m\pi}{3} + \frac{6}{m^2\pi^2}\sin\frac{2m\pi}{3} \\
&= \frac{9}{m^2\pi^2}\sin\frac{2m\pi}{3}.
\end{aligned}
$$

It follows that

$$
A_m = \begin{cases} 0 & \text{if } m \text{ is divisible by } 3 \\ \frac{\sqrt{3}}{2}\frac{9}{m^2\pi^2} & \text{if } m \text{ is congruent to 1 modulo 3} \\ -\frac{\sqrt{3}}{2}\frac{9}{m^2\pi^2} & \text{if } m \text{ is congruent to 2 modulo 3.} \end{cases}
$$

Thus

$$
u(x,0) = \frac{9\sqrt{3}}{2\pi^2}\left(\sin\pi x - \sin 2\pi x + \sin 4\pi x - \sin 5\pi x + \dots\right).
$$

Putting all of this together we get

$$
u(x,t) = x + \frac{9\sqrt{3}}{2\pi^2}\left(e^{-\pi^2 t}\sin\pi x - e^{-4\pi^2 t}\sin 2\pi x + e^{-16\pi^2 t}\sin 4\pi x - e^{-25\pi^2 t}\sin 5\pi x + \dots\right).
$$

5.1.9. Since we have Neumann boundary conditions, we want to take the Fourier cosine series for the initial conditions. $\phi = 0$ and $\psi(x) = \cos^2 x$. Now

$$
\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x,
$$

and so this is the Fourier cosine series for $\psi(x)$. It follows that

$$
B_0 = 1 \qquad B_2 = \frac{1}{2} \qquad \text{and otherwise} \qquad B_m = 0.
$$

On the other hand, $A_m = 0$ as $\phi = 0$. Thus

$$
u(x,t) = \frac{t}{2} + \frac{1}{4c}\sin 2ct\cos 2x,
$$

is the solution to the wave equation, with $u_x(0,t) = u_x(\pi,t) = 0$, $u(x,0) = 0$ and $u_t(x,0) = \cos^2 x$.

5.2.4. (a) Let

$$\phi(x) = \frac{A_0}{2} + \sum_n A_n \cos \frac{n\pi x}{l} + B_n \sin \frac{n\pi x}{l}$$

be the Fourier series of ϕ. If ϕ is an odd function then

$$\phi(x) \cos \frac{m\pi x}{l}$$

is also odd. It follows that

$$\begin{aligned} A_m &= \frac{1}{l} \int_{-l}^{l} \phi(x) \cos \frac{m\pi x}{l} \, \mathrm{d}x \\ &= 0. \end{aligned}$$

Similarly $A_0 = 0$. Thus the Fourier series for ϕ only has sine terms.
(b) If ϕ is an even function then

$$\phi(x) \sin \frac{m\pi x}{l}$$

is an odd function. It follows that

$$\begin{aligned} B_m &= \frac{1}{l} \int_{-l}^{l} \phi(x) \sin \frac{m\pi x}{l} \, \mathrm{d}x \\ &= 0. \end{aligned}$$

Thus the Fourier series for ϕ only has cosine terms.
5.2.6. Let ϕ be a function on $(0, l)$. Let ϕ_{even} be the even extension of ϕ to $(-l, l)$ so that

$$\phi_{\text{even}}(x) = \begin{cases} \phi(x) & \text{if } x > 0 \\ \phi(-x) & \text{if } x < 0. \end{cases}$$

Then the Fourier series for ϕ_{even} only contains cosine terms, so that we have

$$\phi_{\text{even}} = \frac{A_0}{2} + \sum A_n \cos \frac{n\pi x}{l}.$$

If we restrict to $(0, l)$ then the LHS becomes ϕ and we get the Fourier cosine series for ϕ.
5.2.8. (a) Let ϕ be a differentiable function. If $\phi(x)$ is even then

$$\phi'(x) = \lim_{h \to 0} \frac{\phi(x+h) - \phi(x)}{h}.$$

It follows that

$$\begin{aligned}\phi'(-x) &= \lim_{h\to 0} \frac{\phi(-x+h)-\phi(-x)}{h} \\ &= \lim_{h\to 0} \frac{\phi(x-h)-\phi(x)}{h} \\ &= \lim_{g\to 0} \frac{\phi(x+g)-\phi(x)}{-g} \\ &= -\lim_{g\to 0} \frac{\phi(x+g)-\phi(x)}{g} \\ &= -\phi'(x).\end{aligned}$$

Thus $\phi'(x)$ is odd.
On the other hand if ϕ is odd then it follows that

$$\begin{aligned}\phi'(-x) &= \lim_{h\to 0} \frac{\phi(-x+h)-\phi(-x)}{h} \\ &= \lim_{h\to 0} \frac{-\phi(x-h)+\phi(x)}{h} \\ &= \lim_{g\to 0} \frac{-\phi(x+g)+\phi(x)}{-g} \\ &= \lim_{g\to 0} \frac{\phi(x+g)-\phi(x)}{g} \\ &= \phi'(x).\end{aligned}$$

Thus $\phi'(x)$ is even.
(b) Suppose that ϕ is integrable and let

$$\Phi(x) = \int_0^x \phi(s)\,\mathrm{d}s.$$

If ϕ is even then

$$\begin{aligned}\Phi(-x) &= \int_0^{-x} \phi(s)\,\mathrm{d}s \\ &= -\int_0^{x} \phi(-t)\,\mathrm{d}t \\ &= -\int_0^{x} \phi(t)\,\mathrm{d}t \\ &= -\Phi(x),\end{aligned}$$

where to get from the first line to the second line we made the change of variables. Thus Φ is odd.

If ϕ is odd then

$$\begin{aligned}\Phi(-x) &= \int_0^{-x} \phi(s)\,\mathrm{d}s \\ &= -\int_0^{x} \phi(-t)\,\mathrm{d}t \\ &= \int_0^{x} \phi(t)\,\mathrm{d}t \\ &= \Phi(x),\end{aligned}$$

where to get from the first line to the second line we made the change of variables. Thus Φ is even.

5.2.9. The odd coefficients are all zero.

If

$$\phi(x) = \sum_n a_n \sin nx$$

then $\phi(x)$ is odd. Thus its Fourier series over $(-\pi/2, \pi/2)$ is equal to its Fourier sine series over $(0, \pi/2)$.

Thus

$$\phi(x) = \sum_n b_n \sin 2nx.$$

It follows that

$$\sum_n a_n \sin nx = \sum_n b_n \sin 2nx.$$

If we integrate against $\sin mx$, where m is odd, the RHS is zero and so $a_m = 0$ if m is odd.

Challenge Problems: (Just for fun)

5.2.15.

$$|\sin x|$$

is an even function. Thus the Fourier series only invovles cosine terms.