MODEL ANSWERS TO THE SEVENTH HOMEWORK

3.3.1. Let f_{odd} and ϕ_{odd} be the odd extensions of f and ϕ to the whole real line. Let v be the solution to the inhomogeneous diffusion equation on the whole line with source f_{odd} and initial condition ϕ_{odd},

$$v_t - kv_{xx} = f_{\text{odd}}(x,t) \qquad \text{for} \qquad -\infty < x < \infty, \quad 0 < t < \infty,$$

where

$$v(x,0) = \phi_{\text{odd}}(x).$$

Then

$$v(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi_{\text{odd}}(y)\,\mathrm{d}y + \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s) f_{\text{odd}}(y,s)\,\mathrm{d}y\,\mathrm{d}s$$

Let u be the restriction of v to the half line $0 < x < \infty$.
Consider $v(x,t) + v(-x,t)$. This is a solution to the homogeneous diffusion equation with zero initial conditions. Uniqueness implies that $v(x,t) + v(-x,t) = 0$. It follows that $v(x,t)$ is odd and so $u(0,t) = 0$. u is a solution to the inhomogeneous diffusion equation with source f and initial condition $\phi(x)$.
We have

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} S(x-y,t)\phi_{\text{odd}}(y)\,\mathrm{d}y + \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s) f_{\text{odd}}(y,s)\,\mathrm{d}y\,\mathrm{d}s \\ &= \int_0^{\infty} S(x-y,t)\phi(y)\,\mathrm{d}y + \int_{-\infty}^{0} S(x-y,t) - \phi(-y)\,\mathrm{d}y + \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s) f_{\text{odd}}(y,s) \\ &= \int_0^{\infty} (S(x-y,t) - S(x+y,t)\phi(y)\,\mathrm{d}y + \int_0^t \int_0^{\infty} (S(x-y,t-s) - S(x+y,t-s)) f(y,s) \end{aligned}$$

3.3.3. Let $W(x,t) = w(x,t) - xh(t)$. Then

$$W_x = w_x - h(t) \qquad \text{and} \qquad W_t = w_x - xh'(t).$$

Thus

$$W_t - kW_{xx} = -xh'(t).$$

We also have

$$W_x(0,t) = 0 \qquad \text{and} \qquad W(x,0) = \phi(x).$$

Thus W satisfies the inhomogeneous diffusion equation on the half line with source $-xh'(t)$ and initial condition $\phi(x)$.

By 3.3.1 we have

$$W(x,t) = \int_0^\infty (S(x-y,t) - S(x+y,t)\phi(y)\,\mathrm{d}y + \int_0^t \int_0^\infty (S(x-y,t-s) - S(x+y,t-s))xh'(t)\,\mathrm{d}y\,\mathrm{d}s.$$

It follows that

$$w(x,t) = \int_0^\infty (S(x-y,t) - S(x+y,t)\phi(y)\,\mathrm{d}y + \int_0^t \int_0^\infty (S(x-y,t-s) - S(x+y,t-s))xh'(t)\,\mathrm{d}y\,\mathrm{d}s + xh$$

3.4.1. We simply apply the formula. We have

$$\phi(x) = \psi(x) = 0 \qquad \text{and} \qquad f(x,t) = xt$$

so that

$$\begin{aligned} u(x,t) &= \frac{1}{2c} \iint_\Delta f \\ &= \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} ys\,\mathrm{d}y\,\mathrm{d}s \\ &= \frac{1}{4c} \int_0^t \Big[y^2 s \Big]_{x-c(t-s)}^{x+c(t-s)} \mathrm{d}s \\ &= \frac{1}{4c} \int_0^t s\left((x + c(t-s))^2 - (x - c(t-s))^2\right) \mathrm{d}s \\ &= x \int_0^t st - s^2\,\mathrm{d}s \\ &= x\left(\frac{t^3}{2} - \frac{t^3}{3}\right) \\ &= \frac{xt^3}{6}. \end{aligned}$$

3.4.3. Again, we simply apply the formula.

There are three parts. The part corresponding to ϕ is

$$\frac{1}{2}\phi(x+ct) + \frac{1}{2}\phi(x-ct) = \frac{1}{2}\sin(x+ct) + \frac{1}{2}\sin(x-ct).$$

The part corresponding to ψ is

$$\begin{aligned} \frac{1}{2c}\int_{x-ct}^{x+ct} \psi = \frac{1}{2c}\int_{x-ct}^{x+ct} (1+s)\,\mathrm{d}s \\ &= \frac{1}{2c}\left[s + \frac{s^2}{2}\right]_{x-ct}^{x+ct} \\ &= t + \frac{1}{4c}\left((x+ct)^2 - (x-ct)^2\right) \\ &= t + xt. \end{aligned}$$

Finally there is the part corresponding to f

$$\frac{1}{2c}\iint_\Delta f = \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} \cos y \,\mathrm{d}y\,\mathrm{d}s$$
$$= \frac{1}{2c}\int_0^t \Big[\sin y\Big]_{x-c(t-s)}^{x+c(t-s)} \mathrm{d}s$$
$$= \frac{1}{2c}\int_0^t \sin(x+c(t-s)) - \sin(x-c(t-s))\,\mathrm{d}s$$
$$= \frac{1}{2c^2}\Big[\cos(x+c(t-s)) + \cos(x-c(t-s))\Big]_0^t$$
$$= \frac{1}{c^2}\cos x - \frac{1}{2c^2}\left(\cos(x+ct)+\cos(x-ct)\right).$$

Putting all of this together gives

$$u(x,t) = \frac{1}{2}\sin(x+ct) + \frac{1}{2}\sin(x-ct) + t + xt + \frac{1}{c^2}\cos x - \frac{1}{2c^2}\left(\cos(x+ct)+\cos(x-ct)\right).$$

4.1.2. We have the PDE

$$u_t = k u_{xx}$$

where $u(x,0) = 1$ and $u(0,t) = u(l,t) = 0$.
The general solution is

$$u_n(x,t) = \sum_n A_n e^{-\frac{n^2\pi^2}{l^2}kt} \sin\frac{n\pi x}{l}.$$

This satisfies the initial condition

$$u_n(x,0) = \sum_n A_n \sin\frac{n\pi x}{l}.$$

If we compare this with the series expansion

$$1 = \frac{4}{\pi}\left(\sin\frac{\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + \frac{1}{5}\sin\frac{5\pi x}{l} + \dots\right),$$

we see that we want to choose

$$A_n = \begin{cases} \frac{4}{\pi}\frac{1}{2n+1} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Thus

$$u(x,t) = \frac{4}{\pi}\left(e^{-\frac{\pi^2}{l^2}kt}\sin\frac{\pi x}{l} + \frac{1}{3}e^{-9\frac{\pi^2}{l^2}kt}\sin\frac{3\pi x}{l} + \frac{1}{5}e^{-25\frac{\pi^2}{l^2}kt}\sin\frac{5\pi x}{l} + \dots\right)$$

is the heat distribution in the metal rod.
4.1.3. Let

$$u(x,t) = X(x)T(t)$$

be a separated solution. Then

$$T'(t)X(x) = iT(t)X''(x).$$

It follows that

$$\frac{T'}{iT} = \frac{X''}{X} = -\lambda$$

is constant. We have already seen that this implies that

$$X_n(x) = \sin\frac{n\pi x}{l}$$

for some n, where

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2.$$

The equation for T is then

$$T' + i\lambda_n T = 0.$$

This gives

$$T(t) = A_n e^{-i\lambda_n t}.$$

Thus

$$u(x,t) = \sum_n A_n e^{-n^2\pi^2 it/l^2} \sin\frac{n\pi x}{l}.$$

4.1.4. As usual, we look for separated solutions. The PDE becomes

$$\frac{T'' + rT'}{c^2 T} = \frac{X''}{X} = -\lambda.$$

As usual we get

$$X_n(x) = \sin\frac{n\pi x}{l} \qquad \text{where} \qquad \lambda_n = \left(\frac{n\pi x}{l}\right)^2.$$

Thus

$$T'' + rT' + \lambda_n c^2 T = 0.$$

Consider the quadratic equation

$$z^2 + rz + \lambda_n c^2 = 0.$$

The roots are given by the quadratic formula. They are

$$\begin{aligned} z &= \frac{-r \pm \sqrt{r^2 - \lambda_n c^2}}{2} \\ &= \frac{-r \pm i\sqrt{\lambda_n c^2 - r^2}}{2} \\ &= -\frac{r}{2} \pm b_n i. \end{aligned}$$

It follows that

$$T_n(t) = e^{-rt/2}(A_n \cos b_n t + B_n \sin b_n t).$$

Thus we get

$$u(x,t) = e^{-rt/2} \sum_n (A_n \cos b_n t + B_n \sin b_n t) \sin \frac{n\pi x}{l}.$$

Challenge Problems: (Just for fun)

3.4.6. (a) Recall that we have a factorisation

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u \\ &= f(x,t). \end{aligned}$$

If we let

$$v = u_t + cu_x \qquad \text{then} \qquad v_t - cv_x = f.$$

(b) If we introduce the change of coordinates

$$\xi = cx + t \qquad \text{and} \qquad \eta = x - ct$$

then the first equation reduces to

$$u_\xi = v.$$

If we integrate with respect to ξ then we get

$$u(\xi, \eta) = \int^\xi v.$$

Now consider changing back to x, t-coordinates. The integral is over the curves where η is constant. These are parametrised as

$$(x - ct + cs, s).$$

Therefore we get

$$u(x,t) = \int_0^t v(x - ct + cs, s)\, \mathrm{d}s.$$

(c) If we introduce the change of coordinates

$$\xi = cx + t \qquad \text{and} \qquad \eta = x - ct$$

then the first equation reduces to

$$v_\eta = f.$$

If we integrate with respect to ξ then we get

$$v(\xi, \eta) = \int^\eta f.$$

Now consider changing back to x, t-coordinates. The integral is over the curves where ξ is constant. These are parametrised as

$$(x + ct - cr, r).$$

Therefore we get

$$v(x,t) = \int_0^t f(x + ct - cr, r)\,\mathrm{d}r.$$

(d) We have

$$\begin{aligned} u(x,t) &= \int_0^t v(x - ct + cs, s)\,\mathrm{d}s \\ &= \int_0^t \int_0^s f(x - ct + 2cs - cr, r)\,\mathrm{d}r\,\mathrm{d}s \\ &= \int_0^t \int_r^t f(x - ct + 2cs - cr, r)\,\mathrm{d}s\,\mathrm{d}r. \end{aligned}$$

To get from the first line to the second line we used (c) to substitute for v and to get from the second line to the third line we changed the order of integration. Now consider the change of variable $y = x - ct + 2cs - cr$. We get

$$\mathrm{d}y = 2c\,\mathrm{d}s.$$

Thus we get

$$u(x,t) = \frac{1}{2c}\int_0^t \int_{x-ct+cr}^{x+ct-cr} f(y,r)\,\mathrm{d}y\,\mathrm{d}r.$$

3.4.12. We integrate over the reflection Δ of the domain of dependence. There are two cases. If $x_0 + ct_0 \geq 0$ then the domain of dependence has the standard triangular shape and we will get the same answer as before.

If $x_0 + ct_0 < 0$ then the reflection of the domain of dependence is a triangle with a triangle removed (another domain of dependence with vertex along the t-axis). This domain of has four sides, call them M_0, M_1, M_2 and M_3. Compared with the labelling on page 76, M_0 is a part of L_0, $M_1 = L_1$, M_2 is a part of L_2 and M_3 starts on the t-axis and goes down to the x-axis, parallel to M_1.

Note that the line $x - x_0 = c(t - t_0)$ meets the t-axis where

$$t = t_0 - \frac{x_0}{c}.$$

On the other hand the endpoint of M_3 on the x-axis is the reflection of $(x_0 - ct_0, 0)$ which is $(ct_0 - x_0, 0)$.

We have
$$\iint_\Delta f\,\mathrm{d}x\,\mathrm{d}t = \iint_\Delta (v_{tt} - c^2 v_{xx})\,\mathrm{d}x\,\mathrm{d}t.$$
Green's Theorem states that
$$\iint_\Delta (P_x - Q_y)\,\mathrm{d}x\,\mathrm{d}t = \iint_{\partial\Delta} P\,\mathrm{d}t + Q\,\mathrm{d}x,$$
for any functions P and Q. Thus
$$\iint_\Delta f\,\mathrm{d}x\,\mathrm{d}t = \iint_{M_0+M_1+M_2+M_3} -c^2 v_x\,\mathrm{d}t - v_t\,\mathrm{d}x.$$
For the integral over M_0 we have $\mathrm{d}t = 0$ and so
$$\int_{M_0} = -\int_{ct_0-x_0}^{x_0+ct_0} \psi(x)\,\mathrm{d}x.$$
Note the lower limit is now $ct_0 - x_0$. The integral over $M_1 = L_1$ is the same as before,
$$\int_{M_1} = cv(x_0, t_0) - c\phi(x_0 + ct_0).$$
On M_2 we have $x - ct = x_0 - ct_0$ so that $\mathrm{d}x - c\mathrm{d}t = 0$. But then
$$-c^2 v_x\,\mathrm{d}t - v_t\,\mathrm{d}x = -cv_x\,\mathrm{d}x - cv_t\,\mathrm{d}t = -c\,\mathrm{d}v.$$
It follows that
$$\int_{M_2} = cv(x_0, t_0) - ch(t_0 - \frac{x_0}{c}).$$
For the integral over M_3, we have $x+ct = ct_0-x_0$, so that $\mathrm{d}x+c\mathrm{d}t = 0$. But then
$$-c^2 v_x\,\mathrm{d}t - v_t\,\mathrm{d}x = cv_x\,\mathrm{d}x + cv_t\,\mathrm{d}t = c\,\mathrm{d}v.$$
Thus
$$\int_{M_3} = c\int_{M_3} \mathrm{d}v = -ch(t_0 - \frac{x_0}{c}) + cv\phi(ct_0 - x_0).$$
Adding these results gives
$$\iint_\Delta f\,\mathrm{d}x\,\mathrm{d}t = 2cv(x_0, t_0) - 2ch(t_0 - \frac{x_0}{c}) - c\left(\phi(x_0 + ct_0) - \phi(ct_0 - x_0)\right) - \int_{ct_0-x_0}^{x_0+ct_0} \psi(x)\,\mathrm{d}x.$$
Thus
$$v(x_0, t_0) = h(t_0 - \frac{x_0}{c}) + \frac{1}{2}\left(\phi(x_0 + ct_0) - \phi(ct_0 - x_0)\right) + \frac{1}{2c}\int_{ct_0-x_0}^{x_0+ct_0} \psi(x)\,\mathrm{d}x + \frac{1}{2c}\iint_\Delta f\,\mathrm{d}x\,\mathrm{d}t.$$
4.1.6. If we look for separated solutions
$$u(x, t) = X(x)T(t)$$

then we get

$$tXT' = X''T + 2XT.$$

This gives

$$\frac{tT'}{T} - 2 = \frac{X''}{X} = -\lambda.$$

Suppose that $-\lambda = \beta^2 > 0$. For X we get

$$X(x) = C\cos\beta x + D\sin\beta x.$$

As usual, imposing the boundary condition reduces this to

$$X(x) = \sin nx.$$

This gives the ODE

$$tT' = \left(2 - n^2\right)T.$$

The general solution of this ODE is

$$T = At^{2-n^2},$$

where A is a constant. Thus the separated solutions are

$$u_n(x,t) = At^{2-n^2}\sin nx.$$

If we take $n = 1$ then we get

$$u_1(x,t) = At\sin x.$$

This all satisfy the initial condition $u(x,0) = 0$ and so there are infinitely many solutions, one for each value of A.