MODEL ANSWERS TO THE SIXTH HOMEWORK

3.1.1. We use the general formula

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty (e^{-(x-y)^2/4kt} - e^{-(x+y)^2/4kt}) e^{-y} \, \mathrm{d}y.$$

We complete the square in both integrals, as in lecture 11. Note that

$$-(x+y)^2 = -(-(-x)-y)^2.$$

The exponents of the exponentials in the two integrals are

$$-\frac{(y+2kt-x)^2}{4kt} + kt - x \qquad \text{and} \qquad -\frac{(y+2kt+x)^2}{4kt} + kt + x.$$

To get the second expression just flip the sign of x. We make the change of variables

$$p = \frac{y+2kt-x}{\sqrt{4kt}} \qquad \text{and} \qquad q = \frac{y+2kt+x}{\sqrt{4kt}},$$

so that

$$\mathrm{d}p = \frac{\mathrm{d}y}{\sqrt{4kt}} \qquad \text{and} \qquad \mathrm{d}q = \frac{\mathrm{d}y}{\sqrt{4kt}}.$$

Thus

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{\pi}} e^{kt-x} \int_{(2kt-x)/\sqrt{4kt}}^\infty e^{-p^2} \, \mathrm{d}p - \frac{1}{\sqrt{\pi}} e^{kt+x} \int_{(2kt+x)/\sqrt{4kt}}^\infty e^{-q^2} \, \mathrm{d}q \\ &= \frac{1}{2}(e^{kt-x} - e^{kt+x}) - e^{kt-x} \, \mathscr{E}\mathrm{rf}((2kt-x)/\sqrt{4kt}) + e^{kt+x} \, \mathscr{E}\mathrm{rf}((2kt+x)/\sqrt{4kt}). \end{aligned}$$

3.1.3. We want to solve the Neumann boundary problem

$$\begin{aligned} w_t = kw_{xx} \qquad &\text{for} \qquad 0 < x < \infty, \quad 0 < t < \infty \\ w(x,0) = \phi(x) \qquad &\text{for} \qquad t = 0 \\ w_x(0,t) = 0 \qquad &\text{for} \qquad x = 0. \end{aligned}$$

Let ϕ_{even} be the unique function which is the same as ϕ for $x > 0$ (ϕ_{even} extends ϕ) and which is also even.

$$\phi_{\text{even}}(x) = \begin{cases} \phi(x) & \text{if } x > 0 \\ \phi(-x) & \text{if } x < 0. \end{cases}$$

Now we solve the auxiliary problem

$$\begin{aligned} u_t &= k u_{xx} \qquad \text{for} \qquad -\infty < x < \infty, \quad 0 < t < \infty \\ u(x,0) &= \phi_{\text{even}}(x) \qquad \text{for} \qquad t = 0. \end{aligned}$$

We have a formula for

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi_{\text{even}}(y)\,\mathrm{d}y.$$

u is even as ϕ_{even} is even. It follows that u_x is odd. Let $w(x,t)$ be the restriction of $u(x,t)$ to the half line $0 < x < \infty$. Note that as u_x is odd, $w_x(0,t) = 0$. As derivatives are computed locally, w_t is the restriction of u_t and w_{xx} is the restriction of u_{xx}. Thus w automatically satisfies the diffusion equation. As ϕ is the restriction of $\phi_{\text{even}}(x)$ it is automatic that $w(x,0) = \phi(x)$.
We have

$$u(x,t) = \int_0^{\infty} S(x-y,t)\phi(y)\,\mathrm{d}y + \int_{-\infty}^{0} S(x-y,t)\phi(-y)\,\mathrm{d}y.$$

If we change variable from $-y$ to y in the second integral, we get

$$u(x,t) = \int_0^{\infty} (S(x-y,t) + S(x+y,t))\phi(y)\,\mathrm{d}y$$

It follows that

$$w(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_0^{\infty} (e^{-(x-y)^2/4kt} + e^{-(x+y)^2/4kt})\phi(y)\,\mathrm{d}y$$

3.2.1. Consider the Neumann boundary problem

$$\begin{aligned} v_{tt} &= c^2 v_{xx} \qquad \text{for} \qquad 0 < x < \infty, \quad 0 < t < \infty \\ v(x,0) &= \phi(x) \quad v_t(x,0) = \psi(x) \qquad \text{for} \qquad t = 0 \\ v_x(0,t) &= 0 \qquad \text{for} \qquad x = 0. \end{aligned}$$

Let ϕ_{even} and ψ_{even} be the even extensions of ϕ and ψ to the whole real line. Let $u(x,t)$ be the solution to the wave equation on the whole real line and let v be the restriction of u to positive values of x. Then u is even so that u_x is odd and so $v_x(0,t) = 0$.
If we apply d'Alembert's formula then we get

$$v(x,t) = \frac{1}{2}\left(\phi_{\text{even}}(x+ct) + \phi_{\text{even}}(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{even}}(y)\,\mathrm{d}y.$$

We now turn this into a formula involving ϕ and ψ. There are two possibilities, depending on the sign of $x - ct$. If $x > c|t|$ then both

$x+ct$ and $x-ct$ are positive and so

$$v(x,t) = \frac{1}{2}\left(\phi(x+ct) + \phi(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,\mathrm{d}y,$$

the usual formula. But now suppose that $x < c|t|$. Then $x - ct$ is negative so that

$$\phi_{\text{even}}(x-ct) = \phi(ct-x).$$

Thus

$$v(x,t) = \frac{1}{2}\left(\phi(x+ct) + \phi(ct-x)\right) + \frac{1}{2c}\int_{0}^{x+ct} \psi(y)\,\mathrm{d}y + \frac{1}{2c}\int_{x-ct}^{0} \psi(-y)\,\mathrm{d}y,$$

If we replace y by $-y$ in the second integral then we get

$$v(x,t) = \frac{1}{2}\left(\phi(x+ct) + \phi(ct-x)\right) + \frac{1}{2c}\int_{0}^{ct-x} \psi(y)\,\mathrm{d}y + \frac{1}{2c}\int_{0}^{x+ct} \psi(y)\,\mathrm{d}y,$$

valid when $x < c|t|$.
3.2.3. At time $t = 0$ we have

$$u(x,0) = f(x) \qquad \text{and} \qquad u_t(x,0) = cf'(x)$$

We apply d'Alembert's formula. There are two cases. If $x > ct$ then

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} cf'(y)\,\mathrm{d}y \\ &= \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2}\left(f(x+ct) - f(x-ct)\right) \\ &= f(x+xt). \end{aligned}$$

But if $x < ct$ then we get

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left(f(x+ct) - f(ct-x)\right) + \frac{1}{2c}\int_{ct-x}^{x+ct} cf'(y)\,\mathrm{d}y \\ &= \frac{1}{2}\left(f(x+ct) - f(ct-x)\right) + \frac{1}{2}\left(f(x+ct) - f(ct-x)\right) \\ &= f(x+ct) - f(ct-x). \end{aligned}$$

3.2.5. We apply d'Alembert's formula. There are two cases. If $x > ct$ then

$$\begin{aligned} u(x,t) &= \frac{1}{2}(1+1) \\ &= 1. \end{aligned}$$

But if $x < ct$ then we get

$$\begin{aligned} u(x,t) &= \frac{1}{2}(1-1) \\ &= 0. \end{aligned}$$

Thus $u(x,t) = H(x-ct)$. The singularity is at $x = ct$.
3.2.9. (a) $c = 1$ so that the characteristic lines are $x - t = -4/3$ and $x + t = 8/3$. So the two relevant intervals are $[-2, -1]$ and $[2, 3]$. Both the first and the second represents two reflections. Thus the general formula is

$$v(x,t) = \frac{1}{2}\phi(x-t+2) + \frac{1}{2}\phi(x+t-2) + \frac{1}{2}\int_{x-t+2}^{x+t-2} \psi(s)\,\mathrm{d}s.$$

As $\phi(x) = x^2(1-x)$ and $x - t + 2 = x + t - 2$, we get

$$\begin{aligned} v(\frac{2}{3}, 2) &= 2^2(1-2/3)/3^2 \\ &= 4/27. \end{aligned}$$

(b) Now the characteristic lines are $x-t = -13/4$ and $x+t = 15/4$. So the two relevant intervals are $[-4, -3]$ and $[3, 4]$. The first represents four reflections and the second represents three reflections. Thus the general formula is

$$v(x,t) = \frac{1}{2}\phi(x-t+4) - \frac{1}{2}\phi(4-x-t) + \frac{1}{2}\int_{x-t+4}^{4-x-t} \psi(s)\,\mathrm{d}s.$$

Now $x - t + 4 = 3/4$ and $4 - x - t = 1/4$ and so we get

$$\begin{aligned} v(\frac{1}{4}, \frac{7}{2}) &= \frac{1}{2}3^2(1-3/4)/4^2 - \frac{1}{2}1(1-1/4)/4^2 - \frac{1}{2}\left[-(1-x)^3/3\right]_{1/4}^{3/4} \\ &= \frac{1}{2}\frac{3^2-3}{4^3} - \frac{1}{6}\frac{3^3-1}{4^3} \\ &= -\frac{1}{48}. \end{aligned}$$

Challenge Problems: (Just for fun)

3.1.4. (a) $v(x,t)$ is a solution of the diffusion equation with initial conditions $v(x,0) = f(x)$.
(b) As the derivative of a solution to the diffusion equation is a solution to the diffusion equation, we have v_x is a solution of the diffusion

equation. By linearity it follows that w is a solution of the diffusion equation. The initial conditions are given by

$$\begin{aligned} w(x,0) &= v_x(x,0) - 2v(x,0) \\ &= f'(x) - 2f(x) \\ &= \begin{cases} 1-2x & \text{for } x>0 \\ -1-2x & \text{for } x<0 \end{cases} \end{aligned}$$

(c) $f(x) - 2f'(x)$ is clearly odd.
(d) As w is a solution of the diffusion equation and $f(x) - 2f'(x)$ is an odd function, it follows that w is odd.
(e) It follows that $w(0,t) = 0$. Thus $v(x,t)$ satisfies the diffusion equation, with initial condition $v(x,0) = x$ and $v_x(0,t) - 2v(0,t) = w(0,t) = 0$. It follows that

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} f(y)\, \mathrm{d}y.$$