MODEL ANSWERS TO THE FIFTH HOMEWORK

2.4.1. Note that $Q(x,t)$ is a solution to the diffusion equation with $Q(x,0) = H(x)$. This jumps from zero to one at $x = 0$. $\phi(x)$ jumps from 0 to 1 at $x = -l$ and then jumps from 1 to 0 at $x = l$.
Now

$$H(x+l) = \begin{cases} 1 & \text{for } x > -l \\ 0 & \text{for } x < -l. \end{cases}$$

This is almost correct, we just want to adjust this function so that it jumps down at $x = l$.

$$H(x-l) = \begin{cases} 1 & \text{for } x > l \\ 0 & \text{for } x < l. \end{cases}$$

Thus

$$\phi(x) = H(x+l) - H(x-l).$$

$Q(x+l,t)$ is a solution to the diffusion equation with initial conditions $H(x+l$ and $Q(x-l,t)$ is a solution to the diffusion equation with initial conditions $H(x-l)$.
It follows that $Q(x+l,t) - Q(x-l,t)$ is a solution to the diffusion equation with initial condition $H(x+l) - H(x-l) = \phi(x)$.
Now

$$Q(x,t) = \frac{1}{2} + \frac{1}{2}\mathscr{E}\text{rf}\left(\frac{x}{\sqrt{4kt}}\right).$$

Thus

$$Q(x+l,t) - Q(x-l,t) = \frac{1}{2}\left(\mathscr{E}\text{rf}\left(\frac{x+l}{\sqrt{4kt}}\right) - \mathscr{E}\text{rf}\left(\frac{x-l}{\sqrt{4kt}}\right)\right).$$

2.4.9. u_{xxx} is a solution to the diffusion equation, as any derivative of a solution is a solution. As $u(x,0) = x^2$, we have $u_x(x,0) = 2x$, $u_{xx}(x,0) = 2$ and $u_{xxx}(x,0) = 0$. By uniqueness, it follows that $u_{xxx}(x,t) = 0$. If we integrate thrice we get

$$u(x,t) = A(t)x^2 + B(t)x + C(t).$$

In this case

$$u_t = A'(t)x^2 + B'(t)x + C'(t) \qquad \text{and} \qquad u_{xx} = 2A(t).$$

As u is a solution of the diffusion equation we get

$$A'(t)x^2 + B'(t)x + C'(t) = 2kA(t).$$

It follows that $A'(t) = B'(t) = 0$ and $C'(t) = 2A(t)$. From the first two equations we deduce that $A(t) = a$ and $B(t) = b$ are constants. If we plug in $t = 0$ we see that $a = 1$ and $b = 0$. From the equation $C'(t) = 2k$ we see that $C(t)2 = 2kt + c$ and from the initial condition we see that $c = 0$.
Thus

$$u(x,t) = x^2 + 2kt$$

is a solution to the diffusion equation such that $u(x,0) = x^2$.
2.4.10. (a) The general formula says that

$$u(x,t) = \frac{1}{2\sqrt{\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} y^2 \, \mathrm{d}y.$$

If we let

$$p = \frac{x-y}{\sqrt{4kt}} \qquad \text{then} \qquad \mathrm{d}p = -\frac{\mathrm{d}y}{\sqrt{4kt}}.$$

and

$$y^2 = (x - \sqrt{4kt}p)^2.$$

So the integral becomes

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} (x - \sqrt{4kt}p)^2 \, \mathrm{d}p.$$

(b) If we expand the square in the integral, we get three terms,

$$x^2 - 2\sqrt{4kt}px + 4ktp^2.$$

Now consider what happens when we integrate. For the first term we can pull out x^2 and the resulting integral

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} \, \mathrm{d}p = 1.$$

It follows that the coefficient of x^2 is one, as expected. As p is odd and e^{-p^2} is even, pe^{-p^2} is odd and so the integral of the second term is zero. Hence the coefficient of x is zero. Thus

$$u(x,t) = x^2 + \frac{4kt}{\sqrt{\pi}} \int_{-\infty}^{\infty} p^2 e^{-p^2} \, \mathrm{d}p.$$

Comparing we must have

$$\int_{-\infty}^{\infty} p^2 e^{-p^2} \, \mathrm{d}p = \frac{\sqrt{\pi}}{2}.$$

2.4.11. (a) Consider $v(x,t) = u(x,t) + u(-x,t)$. By linearity v is a solution of the diffusion equation. We have

$$\begin{aligned} v(x,0) &= u(x,0) + u(-x,0) \\ &= \phi(x) + \phi(-x) \\ &= 0. \end{aligned}$$

Thus v is a solution to the diffusion equation such that $v(x,0)$ is identically zero. Another such function is the function which is identically zero. By uniqueness v is identically zero.
But then

$$u(x,t) + u(-x,t) = 0,$$

so that u is odd.
(b) Consider $v(x,t) = u(x,t) - u(-x,t)$. By linearity v is a solution of the diffusion equation. We have

$$\begin{aligned} v(x,0) &= u(x,0) - u(-x,0) \\ &= \phi(x) - \phi(-x) \\ &= 0. \end{aligned}$$

Thus v is a solution to the diffusion equation such that $v(x,0)$ is identically zero. Another such function is the function which is identically zero. By uniqueness v is identically zero.
But then

$$u(x,t) - u(-x,t) = 0,$$

so that u is even.
(c) For the wave equation we need that both $\phi(x)$ and $\psi(x)$ are odd (respectively even).
Consider $v(x,t) = u(x,t) \pm u(-x,t)$. By linearity v is a solution of the diffusion equation. We have

$$\begin{aligned} v(x,0) &= u(x,0) \pm u(-x,0) \\ &= \phi(x) \pm \phi(-x) \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} v_t(x,0) &= u_t(x,0) \pm u_t(-x,0) \\ &= \psi(x) \pm \psi(-x) \\ &= 0. \end{aligned}$$

Thus v is a solution to the diffusion equation such that both $v(x,0)$ and $v_t(x,0)$ are identically zero. Another such function is the function which is identically zero. By uniqueness v is identically zero.

But then

$$u(x,t) \pm u(-x,t) = 0,$$

so that u is odd (respectively even).

2.4.12. (a)

$$Q(x,t) = \frac{1}{2} + \frac{1}{2}\mathscr{E}\mathrm{rf}\left(\frac{x}{\sqrt{4kt}}\right).$$

(b) We have

$$e^z = 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \cdots + \frac{1}{n!}z^n + \ldots.$$

Thus

$$e^{-y^2} = 1 - y^2 + \frac{1}{2}y^4 - \frac{1}{6}y^6 + \cdots + (-1)^n\frac{1}{n!}y^{2n} + \ldots.$$

It follows that

$$\int e^{-y^2}\,\mathrm{d}y = y - \frac{1}{3}y^3 + \frac{1}{10}y^5 - \frac{1}{42}y^7 + \cdots + (-1)^n\frac{1}{(2n+1)n!}y^{2n+1} + \ldots.$$

It follows that

$$Q(x,t) = \frac{1}{2} + \frac{1}{2}\frac{x}{\sqrt{4kt}} - \frac{1}{6}\left(\frac{x}{\sqrt{4kt}}\right)^3 + \frac{1}{20}\left(\frac{x}{\sqrt{4kt}}\right)^5 + \cdots + (-1)^n\frac{1}{2(2n+1)n!}\left(\frac{x}{\sqrt{4kt}}\right)^{2n+1} + \ldots.$$

(c) We have

$$Q(x,t) \approx \frac{1}{2} + \frac{1}{2}\frac{x}{\sqrt{4kt}}.$$

(d) If x is fixed and t is large then

$$y = \frac{x}{\sqrt{4kt}}$$

is small and then the Taylor series is a good approximation.

2.4.18. Consider the change of variable $y = x - Vt$, $s = t$. We have

$$\delta_x = \delta_y \qquad \text{and} \qquad \delta_s = -V\delta_y + \delta_t$$

It follows that

$$u_t - ku_{xx} + Vu_x = u_s - ku_{yy}.$$

The second equation is the diffusion equation and it has solution

$$u(y,s) = \frac{1}{2\sqrt{\pi ks}}\int_{-\infty}^{\infty} e^{-(y-z)^2/4ks}\phi(z)\,\mathrm{d}z.$$

It follows that

$$u(x,t) = \frac{1}{2\sqrt{\pi kt}}\int_{-\infty}^{\infty} e^{-(x-Vt-y)^2/4kt}\phi(y)\,\mathrm{d}y.$$

2.5.1. Suppose that we solve the wave equation on the whole real line with initial data

$$\phi(x) = 0 \qquad \text{and} \qquad \psi(x) = e^{-x^2}.$$

Then there is no boundary and initially $u(x, 0) = 0$ but $u(x, t) > 0$ for $t > 0$ small. So the maximum is an interior point of any suitable rectangle.

2.5.2. (a) If $u = f(x - at)$ then

$$u_{tt} = a^2 f''(x - at) \qquad \text{and} \qquad u_{xx} = f''(x - at)$$

This gives

$$a^2 f''(x - at) = c^2 f''(x - at).$$

If f'' is not identically zero it follows that $a^2 = c^2$ so that $a = \pm c$. If f'' is identically zero then f must be linear.

(b) If $u = f(x - at)$ then

$$u_t = af'(x - at) \qquad \text{and} \qquad u_{xx} = f''(x - at)$$

This gives

$$af'(x - at) = kf''(x - at).$$

Subsituting for $y = ax - at$ this gives

$$af'(y) = kf''(y).$$

Integrating we get

$$kf'(y) = af(y) + b.$$

This is an inhomogeneous linear ODE for f. $f(y) = by/k$ is a particular solution. The associated homogeneous is

$$kf'(y) = af(y).$$

This has general solution

$$f(y) = e^{ay/k}.$$

Thus the original equation has general solution

$$f(y) = e^{ay/k} + \frac{by}{k}.$$

This shows that a is arbitrary.

Challenge Problems: (Just for fun)

2.4.16. Let

$$u(x, t) = e^{-bt} v(x, t).$$

Then

$$u_t = -be^{-bt} v(x, t) + e^{-bt} v_t(x, t) \qquad \text{and} \qquad v_{xx} = e^{-bt} u_{xx}.$$

It follows that

$$v_t - ku_{xx} = 0$$

and so

$$u(x,t) = \frac{1}{2\sqrt{\pi kt}} e^{-bt} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \phi(y)\,\mathrm{d}y.$$

2.4.17. Let

$$u(x,t) = e^{-bt^3/3} v(x,t).$$

Then

$$u_t = -bt^2 e^{-bt^3/3} v(x,t) + e^{-bt} v_t(x,t) \qquad \text{and} \qquad v_{xx} = e^{-bt^3/3} u_{xx}.$$

It follows that

$$v_t - ku_{xx} = 0$$

and so

$$u(x,t) = \frac{1}{2\sqrt{\pi kt}} e^{-bt^3/3} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \phi(y)\,\mathrm{d}y.$$