MODEL ANSWERS TO THE FOURTH HOMEWORK

2.2.3. By assumption

$$u_{tt} = c^2 u_{xx}.$$

(a) If $v(x,t) = u(x-y,t)$ then

$$\begin{aligned} v_{tt} &= u_{tt}(x-y,t) \\ &= c^2 u_{xx}(x-y,t) \\ &= c^2 v_{xx}. \end{aligned}$$

(b) If $v = u_x$ then

$$\begin{aligned} v_{tt} &= u_{ttx} \\ &= u_{xtt} \\ &= c^2 u_{xxx} \\ &= c^2 v_{xx}. \end{aligned}$$

(c) If $v = u(ax, at)$ then

$$\begin{aligned} v_{tt} &= a^2 u_{tt}(ax,at) \\ &= a^2 c^2 u_{xx}(ax,at) \\ &= c^2 a^2 u_{xx}(ax,at) \\ &= c^2 v_{xx}. \end{aligned}$$

2.2.5. The relevant PDE is

$$\rho u_{tt} - T u_{xx} + r u_t = 0,$$

where r is a positive constant. We write down the derivative of the kinetic energy and of the potential energy with respect to time and we try to compare them. We have

$$\mathrm{KE} = \frac{1}{2}\rho \int u_t^2 \, \mathrm{d}x \qquad \text{and} \qquad \mathrm{PE} = \frac{1}{2} T \int u_x^2 \, \mathrm{d}x.$$

We have

$$\begin{aligned}\frac{\mathrm{d}\,\mathrm{KE}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\rho\int u_t^2\,\mathrm{d}x\right)\\ &= \rho\int u_t u_{tt}\,\mathrm{d}x\\ &= T\int u_t u_{xx}\,\mathrm{d}x - r\int u_t^2\,\mathrm{d}x\\ &= Tu_t u_x - T\int u_{tx}u_x\,\mathrm{d}x - r\frac{\mathrm{d}\,\mathrm{KE}}{\mathrm{d}t}.\end{aligned}$$

Here we differentiated under the integral sign, replaced ρu_{tt} by Tu_{xx} and finally we applied integration by parts.
The first term on the last line vanishes as it is evaluated at the two endpoints ∞ and $-\infty$, where it is zero. The second term is a derivative:

$$u_{tx}u_x = \frac{\mathrm{d}(\frac{1}{2}u_x^2)}{\mathrm{d}t}.$$

Its integral is the derivative of potential energy. Then

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}t} &= \frac{\mathrm{d}(\mathrm{KE}+\mathrm{PE})}{\mathrm{d}t}\\ &= \frac{\mathrm{d}\,\mathrm{KE}}{\mathrm{d}t} + \frac{\mathrm{d}\,\mathrm{PE}}{\mathrm{d}t} - r\frac{\mathrm{d}\,\mathrm{KE}}{\mathrm{d}t}\\ &= -r\frac{\mathrm{d}\,\mathrm{KE}}{\mathrm{d}t}\\ &< 0.\end{aligned}$$

Thus the total energy

$$\mathrm{E} = \frac{1}{2}\int_{-\infty}^{\infty}(\rho u_t^2 + Tu_x^2)\,\mathrm{d}x$$

is deceasing.
2.3.3. (a) The minimum of $u(x,t)$ on the three sides is 0 and $u(x,t)$ is certainly not constant and so by the strong maximum principle $u(x,t) > 0$ for $0 < x < 1$ and $0 < t < \infty$.
(b) We use the hint. Suppose that the maximum occurs at $X(t)$. By what we just proved $0 < X(t) < 1$. We have

$$\mu(t) = u(X(t),t).$$

We now differentiate with respect to t:

$$\begin{aligned}\frac{\mathrm{d}\mu}{\mathrm{d}t} &= \frac{\mathrm{d}X}{\mathrm{d}t} \cdot u_x(X(t),t) + u_t(X(t),t) \\ &= \frac{\mathrm{d}X}{\mathrm{d}t} \cdot 0 + u_t(X(t),t) \\ &\leq 0.\end{aligned}$$

Here we used the fact that u_x is zero at a critical point and the fact that $u_t \leq 0$ by the maximum principle. It follows that $\mu(t)$ is decreasing.
(c)
2.3.4. (a) The minimum of $u(x,t)$ on the three sides is 0 and $u(x,t)$ is certainly not constant and so by the strong maximum principle $u(x,t) > 0$ for $0 < x < 1$ and $0 < t < \infty$.
(b) Let

$$v(x,t) = u(1-x,t).$$

Then

$$v_t = u_t(1-x,t) \qquad \text{and} \qquad v_{xx} = u_{xx}(1-x,t)$$

(as two wrongs make a right). It follows that v is a solution to the diffusion equation. But for the initial condition, we have

$$\begin{aligned}u(1-x,0) &= 4(1-x)(1-(1-x)) \\ &= 4x(1-x) \\ &= u(x,0).\end{aligned}$$

Thus v is a solution to the diffusion equation with the same initial conditions as u. By uniqueness, $v(x,t) = u(x,t)$, so that

$$u(x,t) = u(1-x,t).$$

(c) We multiply the diffusion equation by u

$$\begin{aligned}0 &= 0 \cdot u \\ &= (u_t - ku_{xx})u \\ &= u_t u - ku_{xx}u \\ &= \frac{1}{2}(u^2)_t - (ku_x u)_x + ku_x^2.\end{aligned}$$

If we integrate over the interval $0 < x < 1$ then we get

$$\int_0^1 \frac{1}{2}(u^2)_t \,\mathrm{d}x - \Big[ku_x u\Big]_0^1 + \int_0^1 ku_x^2 \,\mathrm{d}x.$$

The middle term is zero, because of the boundary conditions and so we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \frac{1}{2}u^2\,\mathrm{d}x = -\int_0^1 ku_x^2\,\mathrm{d}x < 0.$$

Note that $u(x,t) > 0$ for $0 < x < 1$ and yet $u(0,t) = u(1,t) = 0$ so that the last integral is not zero. Thus

$$\int_0^1 u^2\,\mathrm{d}x$$

is strictly decreasing.

2.3.7. (a) We may assume that $t \leq T$, since it is enough to prove that $u \leq v$ for T for arbitrarily large.

Look at the difference $w = v - u$. If $h = g - f$ then $h \geq 0$ and w satisfies

$$w_t - kw_{xx} = h.$$

Further $w \geq 0$ at $t = 0$, $x = 0$ and $x = l$.

Suppose that $w < 0$ somewhere. Then the minimum is not on the three sides $t = 0$, $x = 0$ and $x = l$. Then the same is true of $r(x,t) = w - \epsilon x^2$, for some $\epsilon > 0$.

We look for the minimum point (x_0, t_0) of r. It must either be inside the rectangle or along the side $t = T$. As we saw in lectures, this implies that $r_t(x_0, t_0) \leq 0$ (at a critical point we get zero and even if $t_0 = T$ we get the inequality). On the other hand, $r_{xx}(x_0, t_0) \geq 0$ at a minimum.

Thus

$$r_t(x_0, t_0) - r_{xx}(x_0, t_0) \leq 0.$$

Now

$$r_t = w_t \qquad \text{and} \qquad r_{xx} = w_{xx} - 2\epsilon.$$

It follows that

$$\begin{aligned} r_t - kr_{xx} &= w_t - kw_{xx} + 2k\epsilon \\ &= 2k\epsilon \\ &> 0. \end{aligned}$$

As this is not possible, we must have $w \geq 0$ everywhere.

It follows that $u \leq v$.

(b) Let $u(x,t) = (1 - e^{-t})\sin x$. Then

$$u_t = -e^{-t}\sin x = u_{xx} - \sin x$$

Thus $u(x,t)$ is a solution of the PDE

$$u_t - u_{xx} = \sin x.$$

with initial and boundary conditions

$$u(x,0) = \sin x \qquad u(0,t) = 0 \qquad \text{and} \qquad u(l,\pi) = 0.$$

Thus $u \le v$ for $t = 0$, $x = 0$ and $x = \pi$.
It follows that $u \le v$ so that

$$v(x,t) \ge (1 - e^{-t}) \sin x.$$

2.3.8. We multiply the diffusion equation by u

$$\begin{aligned} 0 &= 0 \cdot u \\ &= (u_t - k u_{xx}) u \\ &= u_t u - k u_{xx} u \\ &= \frac{1}{2}(u^2)_t - (k u_x u)_x + k u_x^2. \end{aligned}$$

If we integrate over the interval $0 < x < l$ then we get

$$\int_0^l \frac{1}{2}(u^2)_t \, \mathrm{d}x - \Big[k u_x u \Big]_0^l + \int_0^l k u_x^2 \, \mathrm{d}x.$$

Now the boundary conditions contribute to the middle term. We have

$$\begin{aligned} \Big[k u_x u \Big]_0^l &= k u(l,t) u_x(l,t) - k u(0,t) u_x(0,t) \\ &= k a_l u^2(l,t) - k a_0 u^2(0,t) \\ &\ge 0, \end{aligned}$$

with equality if and only if $u(0,t) = u(l,t) = 0$. By the Robin boundary condition this would also imply that $u_x(0,t) = u_x(l,t) = 0$.
As usual the first term is the derivative of

$$\int_0^1 u^2 \, \mathrm{d}x$$

and so the Robin boundary contributes to the decrease of this integral.

Challenge Problems: (Just for fun)

4. 2.2.6. (a) We have

$$u_{tt} = \alpha(r) f''(t - \beta(r))$$

and

$$u_r = \alpha'(r) f(t - \beta(r)) - \alpha(r)\beta'(r) f'(t - \beta(r))$$

so that

$$u_{rr} = \alpha''(r) f(t-\beta(r)) - 2\alpha'(r)\beta'(r) f'(t-\beta(r)) - \alpha(r)\beta''(r) f'(t-\beta(r)) + \alpha(r)(\beta'(r))^2 f''(t-\beta(r))$$

This gives an ODE for f.

(b) We get

$$\alpha(r) = c^2\alpha(r)(\beta'(r))^2 \quad -2\alpha'(r)\beta'(r) - \alpha(r)\beta''(r) - \frac{n-1}{r}\alpha(r)\beta'(r) = 0 \quad \alpha''(r) + \frac{n-1}{r}\alpha'(r) = 0.$$

The first equation implies that

$$\beta'(r) = \frac{1}{c}.$$

Thus $\beta''(r) = 0$. These equations reduce to

$$2\alpha'(r) + \frac{n-1}{r}\alpha(r) = 0 \qquad \text{and} \qquad \alpha''(r) + \frac{n-1}{r}\alpha'(r) = 0.$$

(c) Separating variables in the first equation we get

$$\frac{1}{\alpha}\mathrm{d}\alpha = -\frac{n-1}{2r}\mathrm{d}r.$$

Integrating both sides, we get

$$\log\alpha = \log r^{(1-n)/2} + C.$$

Thus

$$\alpha = \frac{c}{r^{(n-1)/2}}$$

for some constant c.
Assume that $n \neq 1$.
Plugging this into the second equation gives

$$\frac{c(n-1)(n+1)}{4r^{(n+3)/2}} - \frac{c(n-1)^2}{2r^{(n+3)/2}} = 0.$$

Assuming $c \neq 0$

$$(n+1) - 2(n-1) = 0.$$

This gives

$$n = 3.$$

(d) As

$$\alpha = \frac{c}{r^{(n-1)/2}}$$

if $n = 1$ then α is constant.
5. 2.3.5. (a) Suppose that

$$u(x,t) = -2xt - x^2.$$

Then

$$u_t = -2x \qquad \text{and} \qquad u_{xx} = -2.$$

so that

$$\begin{aligned} u_t - xu_{xx} &= -2x - x(-2) \\ &= 0. \end{aligned}$$

By standard calculus, the maximum is either in the interior, at a critical point, or on the boundary.
The critical points are where both derivatives are zero,

$$-2x = 0 \qquad \text{and} \qquad -2x - 2t = 0.$$

Thus $x = 0$ and $t = 0$.
Note that

$$u(0, 0) = 0.$$

The boundary is enclosed by the four lines

$$x = \pm 2 \qquad t = 0 \qquad \text{and} \qquad t = 1.$$

We get four functions

$$u(-2, t) = 4t-4 \qquad u(2, t) = -4t-4 \qquad u(x, 0) = -x^2 \qquad \text{and} \qquad u(x, 1) = -2x-x^2.$$

The maximum of the first three functions is 0, at $t = 1$, -4 at $t = 0$ and 0 at $x = 0$ but the maximum of the last is 1 and this is achieved at $x = -1$.
So the maximum value of u is 1 and this is achieved at $(-1, 1)$.
(b) At the maximum, we have $u_t(-1, 1) = 2 > 0$ and $u_{xx}(-1, 1) = -2 < 0$ but this does not violate the equality

$$u_t - xu_{xx} = 0,$$

as it would if u were a solution of the diffusion equation.