MODEL ANSWERS TO THE THIRD HOMEWORK

2.1.1. We apply d'Alembert's formula. By assumption

$$\phi(x) = e^x \qquad \text{and} \qquad \psi(x) = \sin x$$

and so we get

$$u(x,t) = \frac{1}{2}(e^{x+ct} + e^{x-ct}) + \frac{1}{2c}(\cos(x - ct) - \cos(x + ct)).$$

After using some standard identities we get

$$u(x,t) = e^x \cosh ct + \frac{1}{c} \sin x \sin ct.$$

2.1.2. We apply d'Alembert's formula. By assumption

$$\phi(x) = \log(1 + x^2) \qquad \text{and} \qquad \psi(x) = 4 + x$$

and so we get

$$\begin{aligned} u(x,t) &= \frac{1}{2}(\log(1 + (x + ct)^2) + \log(1 + (x - ct)^2)) + \frac{1}{2c}(1/2(x + ct)^2 + 4(x + ct) - 1/2(x - ct)^2 \\ &= \frac{1}{2} \log(1 + (x + ct)^2)(1 + (x - ct)^2) + 5t. \end{aligned}$$

2.1.3. The wave speed is

$$c = \sqrt{\frac{T}{\rho}}.$$

The nearest point where the hammer strikes to the flea is at

$$\frac{l}{2} - a.$$

Therefore the distance of the flea from the nearest point from where the hammer strikes is

$$\frac{l}{4} - a.$$

The hammer causes a depression as well as imparts velocity to the string. It follows that there is a wave traveling to the left at speed c. The disturbance reaches the flea after

$$\frac{l - 4a}{4c} = \frac{\rho^{1/2}(l - 4a)}{4T^{1/2}}$$

units of time.

2.1.5. We apply d'Alembert's formula. By assumption

$$\phi(x) = 0 \qquad \text{and} \qquad \psi(x) = \begin{cases} 1 & \text{for } |x| < a \\ 0 & \text{for } |x| > a. \end{cases}$$

Therefore

$$\begin{aligned} u(x,t) &= \frac{1}{2}\phi(x+ct) + \frac{1}{2}\phi(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi \\ &= \frac{1}{2c}\int_{x-ct}^{x+ct} \psi \\ &= \frac{1}{2c}\{\text{length of } (x-ct, x+ct) \cap (-a,a)\}, \end{aligned}$$

since the integral is nothing more than the area under the graph of ψ, and this area is just the length of the part of $(-a, a)$ between $x - ct$ and $x + ct$.

Now suppose that

$$t = k \cdot \frac{a}{2c} \qquad \text{where} \qquad k = 1, 2, 3, 4, 5.$$

We have

$$x - ct = x - k\frac{a}{2}$$

and so

$$u(x,t) = \frac{1}{2c}\{\text{length of } (x - k\frac{a}{2}, x + k\frac{a}{2}) \cap (-a,a)\}.$$

2.1.9. We have

$$\begin{aligned} u_{xx} - 3u_{xt} - 4u_{tt} &= \partial_x^2 u - 3\partial_x\partial_t u - 4\partial_t^2 u \\ &= (\partial_x^2 - 3\partial_x\partial_t - 4\partial_t^2)u \\ &= (\partial_x - 4\partial_t)(\partial_t + \partial_t)u. \end{aligned}$$

Therefore we have to solve

$$v_x - 4v_t = 0 \qquad \text{where} \qquad v = u_x + u_t.$$

The first equation has general solution

$$v(x,t) = h(4x+t),$$

where h is an arbitrary differentiable function of one variable. Therefore we now just need to solve

$$u_x + u_t = h(4x+t).$$

A particular solution is given by

$$u(x,t) = f(4x+t),$$

where

$$f' = h/5.$$

The associated homogeneous equation is

$$u_x + u_t = 0.$$

This has general solution

$$u(x,t) = g(x-t).$$

Thus the general solution to the original inhomogeneous equation is

$$u(x,t) = f(4x+t) + g(x-t),$$

where f and g are arbitrary twice differentiable functions. It is expedient to make a slightly different choice of f and rewrite this as

$$u(x,t) = f(x+t/4) + g(x-t).$$

We now want to choose f and g such that

$$f(x) + g(x) = x^2 \qquad \text{and} \qquad f'(x)/4 - g'(x) = e^x.$$

Differentiating the first and replacing x by y, we get

$$f'(y) + g'(y) = 2y \qquad \text{and} \qquad f'(y)/4 - g'(y) = e^y.$$

Adding and subtracting we get

$$\frac{5}{4}f'(y) = 2y + e^y \qquad \text{and} \qquad 5g(y) = 2y - 4e^y.$$

It follows that

$$f(y) = \frac{4}{5}\left(y^2 + e^y\right) \qquad \text{and} \qquad g(y) = \frac{1}{5}\left(y^2 - 4e^y\right).$$

Substituting for $4x+t$ and $x-t$ we get

$$f(x+t/4) = \frac{4}{5}\left((x+t/4)^2 + e^{x+t/4}\right) \qquad \text{and} \qquad g(x-t) = \frac{1}{5}\left((x-t)^2 - 4e^{x-t}\right).$$

Thus the solution to the PDE with auxiliary conditions is

$$\begin{aligned} u(x,t) &= \frac{4}{5}\left((x+t/4)^2 + e^{x+t/4}\right) + \frac{1}{5}\left((x-t)^2 - 4e^{x-t}\right) \\ &= \frac{4}{5}\left(e^{x+t/4} - e^{x-t}\right) + x^2 + \frac{1}{4}t^2. \end{aligned}$$

Challenge Problems: (Just for fun)

2.1.8. (a) Let $v = ru$. Then

$$v_r = ru_r + u \qquad \text{and so} \qquad v_{rr} = ru_{rr} + 2u_r.$$

On the other hand

$$v_{tt} = ru_{tt}.$$

It follows that

$$\begin{aligned} v_{tt} &= r u_{tt} \\ &= c^2 \left(r u_{rr} + 2 u_r \right) \\ &= c^2 v_{rr}. \end{aligned}$$

(b) The general solution of the wave equation for v is

$$v(r,t) = f(r+ct) + g(r-ct)$$

where f and g are two arbitrary twice differentiable functions of one variable. It follows that the general solution of the spherical wave equation is

$$u(x,t) = r f(r+ct) + r g(r-ct),$$

where f and g are two arbitrary twice differentiable functions of one variable.

(c) We use d'Alembert's formula for v. We are given

$$v(r,0) = r\phi(r) \qquad \text{and} \qquad v_t(r,0) = r\psi(r).$$

Thus

$$\begin{aligned} u(r,t) &= \frac{1}{r} v(r,t) \\ &= \frac{1}{2}\phi(r+ct) + \frac{1}{2}\phi(r-ct) + \frac{1}{2rc}\int_{r-ct}^{r+ct} r\psi. \end{aligned}$$