MODEL ANSWERS TO THE SECOND HOMEWORK

1.4.1. Try something of the form $u(x,t) = at + x^2$. This certainly satisfies the initial condition

$$u(x,0) = x^2.$$

We have

$$u_t = a \qquad \text{and} \qquad u_{xx} = 2.$$

Thus

$$u(t,x) = x^2 - 2t$$

is a solution of the diffusion equation with the correct auxiliary condition.

1.4.4. (a) With the assumption on the constants, the heat equation reduces to

$$u_t = \Delta u + f(x)$$

where

$$f(x) = \begin{cases} 0 & x < \frac{l}{2} \\ H & x > \frac{l}{2}. \end{cases}$$

The steady-state solution satisfies the additional constraint

$$u_t = 0.$$

The heat equation reduces further to

$$u_{xx} = -f(x).$$

We solve this equation on both intervals. Over the interval $0 < x < l/2$ we have the equation

$$u_{xx} = 0$$

and this has solution

$$u(x,t) = ax + b,$$

where a and b are constants to be determined.

Over the interval $l/2 < x < l$ we have the equation

$$u_{xx} = -H.$$

The general solution is

$$u(x,t) = -Hx^2/2 + cx + d,$$

where c and d are constants to be determined.

There are four boundary conditions, what happens at the two endpoints, the condition that both solutions are equal at $l/2$ and the condition that the heat flow matches at $l/2$.
The two endpoints give

$$b = 0 \qquad \text{and} \qquad -\frac{Hl^2}{2} + cl + d = 0.$$

Thus

$$d = \frac{Hl^2}{2} - cl.$$

Matching u at $x = l/2$ gives

$$\frac{al}{2} = -\frac{Hl^2}{8} + \frac{cl}{2} + d.$$

Matching u_x at $x = l/2$ gives

$$a = -\frac{Hl}{2} + c$$

Substituting for a and d gives an equation for c:

$$\frac{cl}{2} - \frac{Hl^2}{4} = -\frac{Hl^2}{8} + \frac{cl}{2} + \frac{Hl^2}{2} - cl.$$

Thus

$$cl = \frac{5Hl^2}{8}.$$

It follows that

$$c = \frac{5Hl}{8}.$$

Thus

$$a = \frac{Hl}{8} \qquad \text{and} \qquad d = -\frac{Hl^2}{8}.$$

The solution is

$$u(x,t) = \begin{cases} \frac{Hlx}{8} & 0 \leq x \leq l/2 \\ -\frac{Hx^2}{2} + \frac{5Hlx}{8} - \frac{Hl^2}{8} & l/2 \leq x. \end{cases}$$

(b) Over the interval $0 \leq x \leq l/2$ the maximum is at $x = l/2$ and the maximum is

$$\frac{Hl^2}{16}.$$

Over the interval $l/2 \leq x \leq l$ the maximum is at $x = 5l/8$ and the maximum is

$$\frac{9Hl^2}{128}.$$

and this is the hottest temperature, so that $x = 5l/8$ is the hottest point.

1.5.1. The general solution to the ODE is

$$u(x) = a\cos x + b\sin x.$$

The boundary conditions give

$$a = 0 \qquad \text{and} \qquad b\sin L = 0.$$

If $\sin L \neq 0$ the second condition implies that $b = 0$ and the solution is unique. But if $\sin L = 0$ the second condition is vacuous and $u(x) = b\sin x$ is a solution for any b.
Now $\sin L = 0$ if and only if $L = n\pi$ is an integer multiple of π. Thus the solution is unique if and only if L is not an integer multiple of π.
1.5.5. The characteristic curve has equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y.$$

Thus the characteristic curves are

$$y = Ce^x$$

and the general solution of the PDE is

$$u(x, y) = f(e^{-x}y),$$

where f is an arbitrary function of one variable.
(a) We want to choose f so that

$$\begin{aligned} x &= u(x, 0) \\ &= f(e^{-x}). \end{aligned}$$

Let $w = e^{-x}$. Then

$$x = -\log w$$

and

$$f(w) = -\log w.$$

This gives the solution

$$u(x, y) = -\log(ye^{-x}).$$

However the logarithm function is not defined at zero. In fact

$$\lim_{t \to 0^+} \log t = -\infty$$

and so this function does not have the correct behaviour along the boundary.
(b) We want to choose f so that

$$\begin{aligned} 1 &= u(x, 0) \\ &= f(e^{-x}). \end{aligned}$$

Let $w = e^{-x}$. Then

$$f(w) = 1.$$

Thus the solution is $u(x, y) = 1$.
1.6.1. (a) We have $a_{11} = 1$, $a_{12} = -2$ and $a_{22} = 1$. It follows that

$$\begin{aligned} a_{12}^2 &= 4 \\ &> 1 \\ &= a_{11}a_{22}. \end{aligned}$$

Thus we have a hyperbolic PDE.
(b) We have $a_{11} = 9$, $a_{12} = 3$ and $a_{22} = 1$. It follows that

$$\begin{aligned} a_{12}^2 &= 9 \\ &= a_{11}a_{22}. \end{aligned}$$

Thus we have a parabolic PDE.
1.6.2. We have $a_{11} = 1 + x$, $a_{12} = xy$ and $a_{22} = -y^2$. It follows that we have a parabolic PDE when

$$\begin{aligned} x^2y^2 &= a_{12}^2 \\ &= a_{11}a_{22} \\ &= -(1 + x)y^2. \end{aligned}$$

It follows that either $y = 0$ or $x^2 + x + 1 = 0$. As

$$1^2 < 4$$

there are no real solutions to the second equation and we have a parabolic PDE if and only if $y = 0$. If $y \neq 0$ then $a_{12}^2 > a_{11}a_{22}$ and so we have a hyperbolic PDE. 2 1.6.4. We have $a_{11} = 1$, $a_{12} = -2$ and $a_{22} = 4$. It follows that

$$\begin{aligned} a_{12}^2 &= 4 \\ &= 4 \\ &= a_{11}a_{22}. \end{aligned}$$

Thus we have a parabolic PDE.
Suppose that

$$u(x, y) = f(y + 2x) + xg(y + 2x).$$

Then

$$u_x = 2f'(y+2x)+g(y+2x)+2xg'(y+2x) \qquad \text{and} \qquad u_y = f'(y+2x)+xg'(y+2x).$$

It follows that

$$\begin{aligned}
u_{xx} &= 4f''(y+2x) + 2g'(y+2x) + 2g'(y+2x) + 4xg''(y+2x) \\
u_{xy} &= 2f''(y+2x) + g'(y+2x) + 2xg''(y+2x) \\
u_{yy} &= f''(y+2x) + xg''(y+2x).
\end{aligned}$$

Thus

$$\begin{aligned}
u_{xx} - 4u_{xy} + 4u_{yy} &= (4 - 4\cdot 2 + 4)f''(y+2x) + (4-4)g'(y+2x) + (4 - 4\cdot 2 + 4)xg''(y+2x) \\
&= 0.
\end{aligned}$$

1.6.5. If we put

$$u = ve^{\alpha x + \beta y}$$

then

$$u_x = \alpha u + v_x e^{\alpha x + \beta y} \qquad \text{and} \qquad u_y = \beta u + v_y e^{\alpha x + \beta y}.$$

It follows that

$$\begin{aligned}
u_{xx} &= \alpha^2 u + 2\alpha v_x e^{\alpha x + \beta y} + v_{xx} e^{\alpha x + \beta y} \\
u_{yy} &= \beta^2 u + 2\beta v_y e^{\alpha x + \beta y} + v_{yy} e^{\alpha x + \beta y}.
\end{aligned}$$

We want to choose α and β so that the coefficients of v_x and v_y are zero:

$$2\alpha - 2 = 0 \qquad \text{and} \qquad 6\beta + 24 = 0.$$

Thus we let

$$\alpha = 1 \qquad \text{and} \qquad \beta = -4.$$

The coefficient of v is then

$$1 + 3\cdot 4^2 - 2 - 24\cdot 4 + 5 = -44.$$

The PDE then reduces to

$$v_{xx} + 3v_{yy} - 44v = 0.$$

If we put

$$y' = \gamma y$$

then

$$\delta_y = \gamma \delta_{y'}.$$

It follows that

$$v_{yy} = \gamma^2 v_{y'y'}.$$

So if we pick $\gamma = \sqrt{3}$ then we reduced to

$$v_{xx} + v_{y'y'} - 44v = 0.$$

Challenge Problems: (Just for fun)

1.4.6. (a) At equilibrium we have $u_t = 0$ and so the heat equation reads

$$u_{xx} = 0.$$

Solving on the interval $0 \leq x \leq L_1$ we have

$$u(x,t) = ax + b$$

and on the interval $L_1 \leq x \leq L_1 + L_2$ we have

$$u(x,t) = cx + d.$$

There are four boundary conditions, what happens at the two end-points, and the condition that both u and the flux are continuous. As the temperature is zero at 0 we have

$$b = u(0,t) = 0.$$

As the temperature is T at $x = L_1 + L_2$ have

$$c(L_1 + L_2) + d = T.$$

As u is continuous at $x = L_1$ we have

$$aL_1 = cL_1 + d$$

As the flux is continuous at $x = L_1$ we must have

$$\kappa_1 a = \kappa_2 c.$$

From the second equation we get

$$d = T - c(L_1 + L_2).$$

Plugging this into the third equation gives

$$T = aL_1 + cL_2.$$

Multiplying through by κ_2 and using the fourth equation gives

$$\kappa_2 T = \kappa_2 a L_1 + \kappa_1 a L_2 = a(\kappa_2 L_1 + \kappa_1 L_2).$$

It follows that

$$a = \frac{\kappa_2 T}{\kappa_2 L_1 + \kappa_1 L_2}.$$

From there we get

$$c = \frac{\kappa_1 T}{\kappa_2 L_1 + \kappa_1 L_2}.$$

It follows that

$$d = \frac{(\kappa_2 - \kappa_1) L_1 T}{\kappa_2 L_1 + \kappa_1 L_2}.$$

Thus

$$u(x,t) = \begin{cases} \frac{\kappa_2 T x}{\kappa_2 L_1 + \kappa_1 L_2} & \text{for } 0 < x < L_1 \\ \frac{\kappa_1 T x}{\kappa_2 L_1 + \kappa_1 L_2} + \frac{(\kappa_2 - \kappa_1) L_1 T}{\kappa_2 L_1 + \kappa_1 L_2} & \text{for } L_1 < x < L_1 + L_2. \end{cases}$$

Note that we are free to replace κ by k in the formulae above, simply by dividing top and bottom of the fractions by $c\rho$.
(b) In this case we have

$$k_2 L_1 + k_1 L_2 = 1 \cdot 3 + 2 \cdot 1 = 5$$

so that

$$u(x,t) = \begin{cases} 2x & \text{for } 0 < x < 3 \\ 4x - 6 & \text{for } 3 < x < 5. \end{cases}$$

1.4.7. (a) By assumption

$$\frac{\partial \vec{v}}{\partial t} + \frac{c_0^2}{\rho_0} \operatorname{grad} \rho = 0.$$

Therefore

$$\begin{aligned} \frac{\partial \operatorname{curl} \vec{v}}{\partial t} &= \operatorname{curl} \frac{\partial \vec{v}}{\partial t} \\ &= -\operatorname{curl} \frac{c_0^2}{\rho_0} \operatorname{grad} \rho \\ &= -\frac{c_0^2}{\rho_0} \operatorname{curl} \operatorname{grad} \rho \\ &= -\frac{c_0^2}{\rho_0} \nabla \times \nabla \rho \\ &= \vec{0}. \end{aligned}$$

But then $\operatorname{curl} \vec{v}$ is constant, so that if it is zero to begin with, it is zero for all time.
(b) We have

$$\begin{aligned} \frac{\partial^2 \vec{v}}{\partial t^2} &= -\frac{c_0^2}{\rho_0} \frac{\partial \operatorname{grad} \rho}{\partial t} \\ &= -\frac{c_0^2}{\rho_0} \operatorname{grad} \frac{\partial \rho}{\partial t} \\ &= c_0^2 \operatorname{grad} \operatorname{div} \vec{v} \\ &= c_0^2 \nabla \cdot \nabla \vec{v} \\ &= c_0^2 \Delta \vec{v} - \Delta \times \Delta \times \vec{v} \\ &= c_0^2 \Delta \vec{v}. \end{aligned}$$

On the other hand,

$$\begin{aligned}
\frac{\partial^2 \phi}{\partial t^2} &= -\rho_0 \frac{\partial \operatorname{div} \vec{v}}{\partial t} \\
&= -\rho_0 \operatorname{div} \frac{\partial \vec{v}}{\partial t} \\
&= c_0^2 \operatorname{div} \operatorname{grad} \phi \\
&= c_0^2 \Delta \phi.
\end{aligned}$$