SECOND MIDTERM
MATH 110A, UCSD, AUTUMN 18

You have 80 minutes.

There are 6 problems, and the total number of points is 70. Show all your work. *Please make your work as clear and easy to follow as possible.*

Name:______________________________

Signature:____________________________

Student ID #:__________________________

Problem	Points	Score
1	15	
2	10	
3	10	
4	15	
5	10	
6	10	
7	15	
8	10	
9	10	
Total	70	

1. (15pts) (i) *Give the definition of the Heaviside step function.*

$$H(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0. \end{cases}$$

(ii) *Write down the kinetic energy of an infinite piece of string with density ρ and tension T.*

$$\text{KE} = \frac{1}{2}\rho \int_{-\infty}^{\infty} u_t^2 \, \mathrm{d}x.$$

(iii) *State the maximum principle (weak or strong, your choice) for the diffusion equation.*

Strong maximum principle: If $u(x,t)$ is a solution to the diffusion equation on the rectangle $0 \leq x \leq l$ and $0 \leq t \leq T$ and $u(x,t)$ achieves its maximum either at an interior point, where $0 < x < l$ and $0 < t < T$ or where $t = T$, then $u(x,t)$ is constant.

2. (10pts) *Find the solution of the PDE*

$$u_{tt} = c^2 u_{xx}$$

subject to the auxiliary conditions $u(x,0) = e^x$ *and* $u_t(x,0) = \sin x$, *where* $u(x,t)$ *depends on* x *and* t.

We apply d'Alembert's formula

$$u(x,t) = \frac{1}{2}(e^{x+ct} + e^{x-ct}) + \frac{1}{2c}(\cos(x-ct) - \cos(x+ct)).$$

After using some standard identities we get

$$u(x,t) = e^x \cosh ct + \frac{1}{c}\sin x \sin ct.$$

(ii) *The midpoint of a piano string of tension* T, *density* ρ *and length* l *is hit by a hammer whose head diameter is* $2a$ *(assume* $a < l/6$*). How long does it take for the disturbance to reach a flea sitting at a distance of* $l/3$ *from one end?*

The solution involves two waves traveling at speeds

$$c = \sqrt{\frac{T}{\rho}}$$

to the left and right. The flea is at at distance of $l/6$ from the centre of the string, so at a distance of

$$\frac{l}{6} - a$$

from the nearest edge of the hammer blow. Therefore the disturbance meets the flea after

$$\frac{l-6a}{6c} = \frac{\sqrt{\rho}(l-6a)}{6\sqrt{T}}$$

units of time.

3. (10pts) (i) *Solve the PDE*

$$u_t = ku_{xx}$$

subject to the auxiliary conditions

$$u(x,0) = 1 \quad \text{for } |x| < l \qquad \text{and} \qquad u(x,0) = 0 \quad \text{for } |x| > l.$$

where $u(x,t)$ *depends on* x *and* t *and your answer involves the error function.*

Note that

$$u(x,0) = H(x+l) - H(x-l).$$

On the other hand, $Q(x+l,t)$ is a solution to the diffusion equation with initial conditions $H(x+l)$ and $Q(x-l,t)$ is a solution to the diffusion equation with initial conditions $H(x-l)$. It follows that $Q(x+l,t) - Q(x-l,t)$ is a solution to the diffusion equation with initial condition $H(x+l) - H(x-l)$. Thus

$$Q(x+l,t) - Q(x-l,t) = \frac{1}{2}\left(\mathscr{E}\mathrm{rf}\left(\frac{x+l}{\sqrt{4kt}}\right) - \mathscr{E}\mathrm{rf}\left(\frac{x-l}{\sqrt{4kt}}\right)\right).$$

(ii) *If* $u(x,t)$ *is a solution of the diffusion equation* $u_t = ku_{xx}$ *then show that the dilated function* $u(\sqrt{a}x, at)$ *is a solution, where* $a > 0$ *is a constant.*

If we apply the chain rule to

$$v(x,t) = u(\sqrt{a}x, at)$$

then we get

$$\begin{aligned} v_t &= \frac{\partial(at)}{\partial t} u_t \\ &= a u_t. \end{aligned}$$

Similarly

$$\begin{aligned} v_x &= \frac{\partial(\sqrt{a}t)}{\partial x} u_x \\ &= \sqrt{a} u_x, \end{aligned}$$

so that

$$\begin{aligned} v_{xx} &= \sqrt{a}\sqrt{a} u_{xx} \\ &= a u_{xx}. \end{aligned}$$

4. (15pts) *Solve*

$$u_t = k u_{xx}$$

where $u(x,0) = x^2$.

u_{xxx} is a solution to the diffusion equation, as any derivative of a solution is a solution. As $u(x,0) = x^2$, we have $u_x(x,0) = 2x$, $u_{xx}(x,0) = 2$ and $u_{xxx}(x,0) = 0$. By uniqueness, it follows that $u_{xxx}(x,t) = 0$. If we integrate thrice we get

$$u(x,t) = A(t)x^2 + B(t)x + C(t).$$

In this case

$$u_t = A'(t)x^2 + B'(t)x + C'(t) \qquad \text{and} \qquad u_{xx} = 2A(t).$$

As u is a solution of the diffusion equation we get

$$A'(t)x^2 + B'(t)x + C'(t) = 2kA(t).$$

It follows that $A'(t) = B'(t) = 0$ and $C'(t) = 2A(t)$. From the first two equations we deduce that $A(t) = a$ and $B(t) = b$ are constants. If we plug in $t = 0$ we see that $a = 1$ and $b = 0$. From the equation $C'(t) = 2k$ we see that $C(t)2 = 2kt + c$ and from the initial condition we see that $c = 0$.
Thus

$$u(x,t) = x^2 + 2kt$$

is a solution to the diffusion equation such that $u(x,0) = x^2$.

5. (10pts) *Show that if u and v are two solutions of the diffusion equation $u_t = ku_{xx}$ and if $u \leq v$ for $t = 0$, $x = 0$ and $x = l$ then $u \leq v$ for $0 \leq t \leq \infty$ and $0 \leq x \leq l$.*

It suffices to show this in the rectangle defined by the extra condition $t \leq T$.
Let $w = v - u$. Note that w is a solution to the diffusion equation by linearity. By hypothesis $w \geq 0$ for for $t = 0$, $x = 0$ and $x = l$. By the minimum principle the minimum of w occurs on the three sides $t = 0$, $x = 0$ and $x = l$. The minimum on these three sides is at least zero and so the minimum of w on the whole rectangle is at least zero, that is, $w \geq 0$ on the whole rectangle.
But then $u \leq v$ on the whole rectangle.

6. (10pts) *Consider the diffusion equation*

$$u_t = k u_{xx}$$

subject to the condition $u(x,0) = \phi(x)$.
Show that if $\phi(x)$ *is odd then* $u(x,t)$ *is an odd function of* x.

Consider $v(x,t) = u(x,t) + u(-x,t)$. By linearity v is a solution of the diffusion equation. We have

$$\begin{aligned} v(x,0) &= u(x,0) + u(-x,0) \\ &= \phi(x) + \phi(-x) \\ &= 0. \end{aligned}$$

Thus v is a solution to the diffusion equation such that $v(x,0)$ is identically zero. Another such function is the function which is identically zero. By uniqueness v is identically zero.
But then

$$u(x,t) + u(-x,t) = 0,$$

so that u is odd.

Bonus Challenge Problems

7. (15pts) *Solve*

$$u_{xx} + u_{xt} - 20u_{tt} = 0$$

where $u(x,0) = \phi(x)$ *and* $u_t(x,0) = \psi(x)$.

$$\begin{aligned} u_{xx} + u_{xt} - 20u_{tt} &= \partial_x^2 u + \partial_x \partial_t u - 20\partial_t^2 u \\ &= (\partial_x - 4\partial_t)(\partial_t + 5\partial_t)u. \end{aligned}$$

Therefore we have to solve

$$v_x - 4v_t = 0 \qquad \text{where} \qquad v = u_x + 5u_t.$$

The first equation has general solution

$$v(x,t) = h(4x + t),$$

where h is an arbitrary differentiable function of one variable. Therefore we now just need to solve

$$u_x + 5u_t = h(4x + t).$$

A particular solution is given by

$$u(x,t) = f(4x + t) \qquad \text{where} \qquad f' = h/9.$$

The associated homogeneous equation

$$u_x + 5u_t = 0 \qquad \text{has general solution} \qquad u(x,t) = g(5x - t).$$

Thus the general solution to the original inhomogeneous equation is

$$u(x,t) = f(x + t/4) + g(x - t/5).$$

We now want to choose f and g such that

$$f(y) + g(y) = \phi(y) \qquad \text{and} \qquad f'(y)/4 - g'(y)/5 = \psi(y).$$

Differentiating the first equation and solving for f' and g' we get

$$f'(y) = \frac{4}{9}(\phi'(y) + 5\psi(y)) \qquad \text{and} \qquad g'(y) = \frac{5}{9}(\phi'(y) - 4\psi(y)).$$

Integrating and substituting for $x + t/4$ and $x - t/5$ we get

$$f(x+t/4) = \frac{4}{9}(\phi(x+t/4) + \int_{-\infty}^{x+t/4} 5\psi(y)\,\mathrm{d}y) \quad \text{and} \quad g(x-t/5) = \frac{5}{9}(\phi(x-t/5) - \int_{-\infty}^{x-t/5} 4\psi(y)\,\mathrm{d}y).$$

Thus the solution to the PDE with auxiliary conditions is

$$u(x,t) = \frac{4}{9}(\phi(x+t/4) + \int_{-\infty}^{x+t/4} 5\psi(y)\,\mathrm{d}y) + \frac{5}{9}(\phi(x-t/5) - \int_{-\infty}^{x-t/5} 4\psi(y)\,\mathrm{d}y).$$

8. (10pts) *Consider the solution $Q(x,t)$ of the diffusion equation such that $Q(x,0) = H(x)$ is the Heaviside step function.*
Assuming that Q is a function g of

$$p = \frac{x}{\sqrt{4kt}}$$

derive an ODE satisfied by g and solve this ODE to find an expression for $Q(x,t)$.

See lecture 11.

9. (10pts) *Use your answer to 8 to find a formula for the solution* $u(x,t)$ *to the diffusion equation such that* $u(x,0) = \phi(x)$.

See lecture 11.