9. Causality and Energy

As writing down explicit solutions to PDEs can be quite hard, it is often just as useful to describe interesting qualitative behaviour. We give two here.

The solution to the IVP for the wave equation has two parts. From the initial position $\phi(x)$ we get two waves traveling to the left and to the right at speed c. The initial velocity $\psi(x)$ also gives rise to a wave traveling at speed $\leq c$. In total we get a wave traveling at speed $\leq c$, which is perhaps lagging behind because of the initial velocity. However, the wave never travels faster than speed c.

We can visualise this in the (x,t)-plane as saying that the wave starting at a point $(x_0, 0)$ never gets out of a shaded region bounded by the characteristic lines $x - x_0 = \pm ct$. This is called the **principle of causality**. The shaded region is called the **domain of influence**.

For example if ϕ and ψ vanish for $|x| > R$ then $u(x,t)$ vanishes for $|x| > R+ct$. Put differently the domain of influence of a sector $|x| \leq R$ is the sector $|x| \leq R + ct$.

Another way to look at the principle of causality is to figure out what could have influenced what happens at a point (x,t). It only depends on the value of ϕ at $x \pm ct$ and on the value of $\psi(x)$ over the interval $[x-ct, x+ct]$. We say that the interval $(x-ct, x+ct)$ is the **domain of influence** at time $t = 0$ on the point (x,t). The entire shaded triangle spanned by characteristic lines is called the **domain of dependence** or the **past history** of the point (x,t).

It is also interesting to be able to write down quantities depending on $u(x,t)$ that are constant in time. From the point of view of physics one of the most important quantities that is constant is the energy.

For example, consider an infinite string with constants the density ρ and the tension T. We have

$$\rho u_{tt} = T u_{xx} \qquad \text{for} \qquad -\infty < x < \infty.$$

The energy in this system has two parts, the kinetic energy and the potential energy. If a particle of mass m is moving with velocity v then the kinetic energy is $1/2mv^2$. Thus the kinetic energy of the string is

$$\mathrm{KE} = \frac{1}{2}\rho \int u_t^2 \, \mathrm{d}x.$$

A few words about convergence: The range of integration is $-\infty$ to ∞. We assume that $\phi(x)$ and $\psi(x)$ are zero for $|x| > R$ so that u and u_t are zero for $|x| > R + ct$ by the principle of causality. Thus the integrals all converge.

Imagine a spring with tension T displaced by an amount y. The potential energy is $1/2Ty^2$. Thus the integral

$$\mathrm{PE} = \frac{1}{2}T\int u_x^2\,\mathrm{d}x.$$

is the potential energy stored in the string. Let

$$\mathrm{E} = \mathrm{KE} + \mathrm{PE},$$

the total energy of the system.

If we differentiate the kinetic energy with respect to time t we get

$$\begin{aligned}
\frac{\mathrm{d}\,\mathrm{KE}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\rho\int u_t^2\,\mathrm{d}x\right) \\
&= \rho\int u_t u_{tt}\,\mathrm{d}x \\
&= T\int u_t u_{xx}\,\mathrm{d}x \\
&= T u_t u_x - T\int u_{tx}u_x\,\mathrm{d}x.
\end{aligned}$$

Here we differentiated under the integral sign, replaced ρu_{tt} by Tu_{xx} and finally we applied integration by parts.

The first term on the last line vanishes as it is evaluated at the two endpoints ∞ and $-\infty$, where it is zero. The second term is a derivative:

$$u_{tx}u_x = \frac{\mathrm{d}(\frac{1}{2}u_x^2)}{\mathrm{d}t}.$$

Its integral is the derivative of potential energy. Then

$$\begin{aligned}
\frac{\mathrm{d}E}{\mathrm{d}t} &= \frac{\mathrm{d}(\mathrm{KE}+\mathrm{PE})}{\mathrm{d}t} \\
&= \frac{\mathrm{d}\,\mathrm{KE}}{\mathrm{d}t} + \frac{\mathrm{d}\,\mathrm{PE}}{\mathrm{d}t} \\
&= 0.
\end{aligned}$$

Thus the total energy

$$\mathrm{E} = \frac{1}{2}\int_{-\infty}^{\infty}(\rho u_t^2 + T u_x^2)\,\mathrm{d}x$$

is constant. This is called the law of **conservation of energy**.

Example 9.1. *Imagine an infinite string with initial position*

$$\phi(x) = \begin{cases} b - \frac{b|x|}{a} & \text{for } |x| < a \\ 0 & \text{for } |x| > a, \end{cases}$$

and initial velocity $\psi(x) = 0$.

We calculate the energy. It suffices to do this at time $t = 0$. In this case the kinetic energy is zero, as the initial velocity is zero. From $-a$ to 0, $u_x = b/a$ and from 0 to a, $u_x = -b/a$ and otherwise $u_x = 0$. Thus $u_x^2 = b^2/a^2$ from $-a$ to a and otherwise it is zero. Thus

$$\begin{aligned} \mathrm{E} &= \mathrm{KE} + \mathrm{PE} \\ &= 0 + \frac{1}{2}T(2a)\frac{b^2}{a^2} \\ &= \frac{Tb^2}{a}. \end{aligned}$$