8. Initial value problem

We now solve the wave equation, subject to some initial conditions. We want to solve

$$u_{tt} = c^2 u_{xx} \qquad \text{for} \qquad -\infty < x < \infty,$$

with the initial conditions

$$u(x,0) = \phi(x) \qquad \text{and} \qquad u_t(x,0) = \psi(x),$$

where ϕ and ψ are arbitrary twice differentiable functions of x.

There is one and only one solution of the IVP. For example if $\phi(x) = \cos x$ and $\psi(x) = 0$ then $u(x,t) = \sin x \cos ct$.

It is not hard to write down the solution to the IVP. We start with the general solution to the wave equation.

$$u(x,t) = f(x+ct) + g(x-ct).$$

If we $t = 0$ we get

$$f(x) + g(x) = \phi(x).$$

If we differentiate with respect to t and set $t = 0$ we get

$$cf'(x) - cg'(x) = \psi(x).$$

If we differentiate the first equation with respect to x we get

$$\begin{aligned} f'(x) + g'(x) &= \phi'(x) \\ cf'(x) - cg'(x) &= \psi(x). \end{aligned}$$

We now have a pair of simultaneous linear equations for $f'(x)$ and $g'(x)$. We can solve this in the usual way. It is expedient to introduce a new variable s and replace x by s. Divide the first equation by c and add or subtract it the first equation to get

$$f'(s) = \frac{1}{2}\left(\phi' + \frac{\psi}{c}\right) \qquad \text{and} \qquad g'(s) = \frac{1}{2}\left(\phi' - \frac{\psi}{c}\right).$$

This gives us two ODEs for f and g.

If we solve these the usual way by integrating we get

$$f(s) = \frac{1}{2}\phi(s) + \frac{1}{2c}\int_0^s \psi + A$$

and

$$g(s) = \frac{1}{2}\phi(s) - \frac{1}{2c}\int_0^s \psi + B,$$

where A and B are constants. If we plug in $s = 0$ then both integrals dissappear. If we then add both equations together we see that $A+B = 0$.

If we substitute $s = x + ct$ into the formula for f and $s = x - ct$ into the formula for g and add the result together we get

$$u(x,t) = \frac{1}{2}\phi(x+ct) + \frac{1}{2c}\int_0^{x+ct}\psi + \frac{1}{2}\phi(x-ct) - \frac{1}{2c}\int_0^{x-ct}\psi.$$

Combining the two integrals gives

$$u(x,t) = \frac{1}{2}\phi(x+ct) + \frac{1}{2}\phi(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi.$$

This solution was obtained by d'Alembert in 1746.

If ϕ has continuous second derivative, $\phi \in \mathcal{C}^2$, ψ has continuous first derivative, $\psi \in \mathcal{C}^2$ then $u(x,t)$ has continuous second derivatives, that is, $u(x,t) \in \mathcal{C}^2$.

Example 8.1. *If $\phi(x) = 0$ and $\psi(x) = e^x$ then*

$$u(x,t) = \frac{1}{2c}(e^{x+ct} - e^{x-ct}).$$

Indeed,

$$u_{tt} = \frac{c}{2}(e^{x+ct} - e^{x-ct})$$

and

$$u_{xx} = \frac{1}{2c}(e^{x+ct} - e^{x-ct}),$$

so that

$$u_{tt} = c^2 u_{xx}.$$

If we plug in $t = 0$ we surely get 0. If we differentiate once and put $t = 0$ the second term is equal to the first term and we get e^x.

Example 8.2 (plucked string)**.** *If we have a string with tension T and density ρ then*

$$c = \sqrt{\frac{T}{\rho}}.$$

Imagine an infinite string with initial position

$$\phi(x) = \begin{cases} b - \frac{b|x|}{a} & \text{for } |x| < a \\ 0 & \text{for } |x| > a, \end{cases}$$

and initial velocity $\psi(x) = 0$. This might represent someone who just plucked a guitar string.

The solution is

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right].$$

The original position is represented by one triangle with two vertices along the x-axis, $x = \pm a$ and one vertex along the y-axis, at the point $(0, b)$. This is in fact the superposition of two triangles, with the same vertices along the x-axis but which only go half way up the y-axis.

As time progresses one triangle moves to the right and one triangle moves to the left. Even though the geometric picture is clear, the actual function is harder to write down. For a start we need to worry about the relationship between 0, $\pm a$ and $x \pm ct$.

For example, suppose that

$$t = \frac{a}{2c}.$$

Then

$$x \pm ct = a \pm \frac{a}{2}.$$

It follows that the triangles are now centred at $\pm a/2$. If you draw a picture then $u(x, t)$ is level from $-a/2$ to $a/2$, at level $b/2$.

Thus

$$u(x,t) = \begin{cases} 0 & \text{if } |x| > 3a/2 \\ \frac{3}{4}b(1 - \frac{2|x|}{3a}) & \text{if } a/2 < |x| < 3a/2 \\ b/2 & \text{if } |x| < a/2. \end{cases}$$