7. The wave equation on the line

We next develop some concrete techniques to solve PDEs.

We start with the one dimensional wave equation on the x-axis. Mathematically this avoids the hard problem of dealing with boundary conditions. Physically this is justified if the time is relatively small in relation to distance to the boundary. It takes time for something happening far away to have any effect.

We are going to look at the following PDE

$$u_{tt} = c^2 u_{xx} \qquad \text{for} \qquad -\infty < x < \infty.$$

(This might represent a very long string). This is the simplest second order equation one might study because we can factor:

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u \\ &= 0. \end{aligned}$$

It follows that solving the wave equation is relatively easy. Given $u(x,t)$, compute

$$v = u_t + cu_x \qquad \text{and then compute} \qquad v_t - cv_x.$$

$u(x,t)$ is a solution of the wave equation if and only if the last expression is zero.

The general solution is

$$u(x,t) = f(x+ct) + g(x-ct),$$

where f and g are two arbitrary (twice differentiable) functions of one variable.

Proof. As already observed, we have

$$u_{tt} = c^2 u_{xx}$$

if and only if

$$v_t - cv_x = 0 \qquad \text{where} \qquad v = u_t + cu_x.$$

The first equation on the previous line has solution

$$v(x,t) = h(x+ct),$$

where h is an arbitrary differentiable function. Hence we are reduced to solving

$$u_t + cu_x = h(x+ct).$$

This is a linear inhomogeneous first order equation with constant coefficients.

It is easy to check that one solution is

$$u(x,t) = f(x+ct)$$

where

$$f'(s) = \frac{h(s)}{2c}.$$

Note that since h is arbitrary then f is arbitrary as well, subject to the condition that f is twice differentiable.

Now the homogeneous PDE

$$u_t + cu_x = 0$$

has solution

$$u(x,t) = g(x-ct)$$

where g is an arbitrary differentiable function. For g to be a solution to the wave equation, g has to be twice differentiable.

It follows that the general solution to the original PDE is

$$u(x,t) = f(x+ct) + g(x-ct),$$

where f and g are arbitrary twice differentiable functions.

Aliter: Consider the change of coordinates

$$\xi = x + ct \qquad \text{and} \qquad \eta = x - ct.$$

By the chain rule, we have

$$\partial_x = \partial_\xi + \partial_\eta \qquad \text{and} \qquad \partial_t = c\partial_\xi + c\partial_\eta.$$

It follows that

$$\partial_t - c\partial_x = -2c\partial_\eta \qquad \text{and} \qquad \partial_t + c\partial_x = 2c\partial_\xi.$$

Hence

$$\begin{aligned}\partial_{tt} - c^2\partial_{xx} &= (\partial_t - c\partial_x)(\partial_t + c\partial_x) \\ &= -4c^2\partial_\eta\partial_\xi.\end{aligned}$$

So the original PDE reduces to the PDE

$$\partial_\eta\partial_\xi u = 0 \qquad \text{that is} \qquad u_{\eta\xi} = 0.$$

We have already seen that this has solution

$$u = f(\xi) + g(\eta),$$

where f and g are arbitrary twice differentiable functions.

Substituting for x and t gives the same solution as before. □

The general solution to the wave equation has a very beautiful geometric and physical description. There are two families of characteristic curves,

$$x \pm ct = C,$$

C a constant. These cover the (x, t)-plane twice. The function $g(x-ct)$ is a wave of arbitrary shape moving to the right at speed c. The function $f(x+ct)$ is a wave of arbitrary shape moving to the left at speed c.