6. Types of second-order equations

We will see how the three equations, the Laplace equation, the wave equation and the heat equation cover the qualitative behaviour of all 2nd order PDEs.

In fact we will see later that the wave equation and Laplace's equation behave quite differently, even though it is simply a difference in parity:

$$u_{xx} - u_{yy} = 0 \qquad \text{and} \qquad u_{xx} + u_{yy} = 0.$$

Contrast this with the behaviour of conics, the solutions to quadratic polynomials. There are three types of conics, ellipses, parabolas and hyperbolas. For example

$$x^2 + y^2 = 1 \qquad \text{and} \qquad x^2 - y^2 = 1$$

represent the equations of a circle and a hyperbola.

The general 2nd order constant coefficient linear PDE has the form

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + a_2u_y + a_0u = 0.$$

Let

$$\mathscr{D} = a_{12}^2 - a_{11}a_{22}$$

be the **discriminant**.

Theorem 6.1. *There is a linear change of the variables x and y so that the equation above has one of three forms:*

(i) ***Elliptic case:*** $\mathscr{D} < 0$

$$u_{xx} + u_{yy} + \cdots = 0.$$

(ii) ***Hyperbolic case:*** $\mathscr{D} > 0$

$$u_{xx} - u_{yy} + \cdots = 0.$$

(iii) ***Parabolic case:*** $\mathscr{D} = 0$

$$u_{xx} + \cdots = 0.$$

where dots indicate first order and constant terms.

Proof. We may assume that $a_0 = a_1 = a_2 = 0$. Possibly replacing x by $x + y$ we may assume that $a_{11} \neq 0$. Possibly rescaling we may assume that $a_{11} = 1$.

Completing the square our equation now reads

$$(\partial_x + a_{12}\partial_y)^2u + (a_{22} - a_{12}^2)\partial_y^2u = 0.$$

In case (i) we let

$$b = (a_{22} - a_{12}^2)^{1/2} > 0.$$

Introduce the new variables ξ and η,

$$x = \xi \qquad \text{and} \qquad y = a_{12}\xi + b\eta.$$

By the chain rule,

$$\partial_\xi = 1 \cdot \partial_x + a_{12}\partial_y \qquad \text{and} \qquad \partial_\eta = 0 \cdot \partial_x + b\partial_y.$$

In the new coordinates the PDE becomes

$$\partial_\xi^2 u + \partial_\eta^2 u = 0.$$

The other cases proceed in a similar fashion. $\square$

Example 6.2. *Classify the following PDEs*

(i) $u_{xx} + 3u_{xy}$;
(ii) $45u_{xx} - 30u_{xy} + 5u_{yy} + 2u_x = 0$;
(iii) $5u_{xx} + 6u_{xy} + 2u_{yy} = 0$.

We just need to find the parity of the discriminant. For (i) we have

$$a_{11} = 1 \qquad a_{12} = 3/2 \qquad \text{and} \qquad a_{22} = 0.$$

Hence

$$\begin{aligned} \mathscr{D} &= (3/2)^2 - 1 \\ &= 5/4 \\ &> 0, \end{aligned}$$

a hyperbolic equation.

For (ii) we have

$$a_{11} = 45 \qquad a_{12} = -15 \qquad \text{and} \qquad a_{22} = 5.$$

Hence

$$\begin{aligned} \mathscr{D} &= (-15)^2 - 45 \cdot 5 \\ &= 3^2 5^2 - 3^2 5^2 \\ &= 0, \end{aligned}$$

a parabolic equation.

Finally for (iii) we have

$$a_{11} = 5 \qquad a_{12} = 3 \qquad \text{and} \qquad a_{22} = 2.$$

Hence

$$\begin{aligned} \mathscr{D} &= 3^2 - 5 \cdot 2 \\ &= 9 - 10 \\ &= -1 \\ &< 0, \end{aligned}$$

an elliptic equation.

It is possible to generalise the notions of bolicity to higher dimensions. For this, one needs to diagonalise a symmetric matrix over the reals and take account of the number of the number of ± 1 on the diagonal.

Perhaps more intriguingly, it is also possible to classify linear, order two equations with non-constant coefficients into elliptic, parabolic and hyperbolic. In this case the classification into types varies as one moves around the (x, y)-plane.

Example 6.3. *Find the regions in the* (x, y)*-plane where the PDE*

$$x^2 u_{xx} + 2(1 - y^2) u_{xy} + u_{yy} = 0$$

is elliptic, parabolic and hyperbolic.

We have

$$a_{11} = x^2 + y^2 \qquad a_{12} = 1 \qquad \text{and} \qquad a_{22} = 1.$$

Hence

$$\begin{aligned} \mathscr{D} &= 1^2 - (x^2 + y^2) \\ &= 1 - x^2 - y^2. \end{aligned}$$

Thus we have a parabolic PDE on the circle $x^2 + y^2 = 1$, a hyperbolic PDE inside the circle, where $x^2 + y^2 < 1$ and an elliptic PDE outside the circle, where $x^2 + y^2 > 1$.