5. Well-posed problems

A well-posed problem involves a PDE together with some auxiliary conditions that enjoy the following three properties:

(i) **Existence:** There is at least one solution $u(x,t)$.
(ii) **Uniqueness:** There is at most one solution $u(x,t)$.
(iii) **Stability:** The unique solution $u(x,t)$ depends in a stable manner on the auxiliary conditions; small changes in the conditions give small changes to $u(x,t)$.

If there are too many auxiliary conditions to be satisfied and sometimes there is no solution then we say the problem is called **overdetermind**. If there are too few auxiliary conditions to be satisfied and sometimes there is one than one solution then we say the problem is called **underdetermind**. In practice one can never completely determin the initial data and so stability is important if you want to predict what is going to happen. Unfortunately, instability (or chaos) is surprisingly common.

The mathematical goal is to find the correct auxiliary conditions to guarantee a well-posed problem.

For example, for the vibrating string, with an external force and endpoints which are free to move up and down, we have

$$Tu_{tt} - \rho u_{xx} = f(x,t)$$

subject to the auxiliary conditions

$$\begin{aligned} u(x,0) &= \phi(x) & \qquad u_t(x,t) &= \psi(x) \\ u(0,t) &= g(t) & \qquad u(L,t) &= h(t). \end{aligned}$$

The input data consists of five functions, $f(x,t)$, $\phi(x)$, $\psi(x)$, $g(t)$ and $h(t)$. Existence and uniqueness means that there is a unique solution for every choice of these five functions. Stability means that if we change these five functions by a small amount then the unique solution changes by a small amount.

In fact, there are many ways to mathematical quantify what it means to change a function by a small amount.

Now suppose we consider the heat equation, say describing the heat distribution in a coffee cup. If we pose some initial conditions then it is clear physically that the problem is well-posed. There is a solution (there is a temperature distribution in the coffee cup), the solution is unique and varies continuously on the input data.

But now suppose that we go backwards in time. If we know the temperature distribution now, can we figure out the temperature distribution in the past? A moments thought makes it clear that we cannot. This problem is not well-posed.

Now consider Laplace's equation

$$u_{xx} + u_{yy} = 0,$$

in the region

$$D = \{\, (x, y) \,|\, y > 0 \,\}$$

the upper half plane. The boundary of this region is the x-axis.

It is not a well-posed problem to specify u and u_y on the boundary. Indeed,

$$u_n(x, y) = \frac{1}{n} e^{-\sqrt{n}} \sin nx \sinh ny$$

are solutions to Laplace's equation. On the boundary we have

$$u_n(x, 0) = 0$$

and

$$\begin{aligned} \frac{\partial u_n}{\partial n}(x, 0) &= -\frac{\partial u_n}{\partial y}(x, 0) \\ &= -e^{-\sqrt{n}} \sin nx. \end{aligned}$$

As x tends to zero this tends to zero. However the solutions $u_n(x, y)$ do not tend to zero, for fixed $y > 0$ as n goes to infinity.