## 4. Initial and Boundary conditions

Since solutions to PDEs are ambiguous by functions of one variable, to get a unique solution we need to impose auxiliary conditions. The type of condition one needs to impose to ensure a unique solution depends heavily on the nature of the PDE. The conditions we impose are motivated by the physics and they come in two varieties, initial conditions and boundary conditions.

An **initial condition** specifies the physical state at a particular time $t_0$. For the diffusion equation, the initial condition is

$$u(\vec{x}, t_0) = \phi(\vec{x}),$$

for some function $\phi(\vec{x}) = \phi(x, y, z)$, which represents the initial concentration of dye. We have the same setup for the heat equation, and in this case $\phi(\vec{x})$ is the initial temperature distribution.

For the wave equation we need to specify a pair of initial conditions

$$u(\vec{x}, t_0) = \phi(\vec{x}) \qquad \text{and} \qquad u_t(\vec{x}, t_0) = \psi(\vec{x}).$$

Here $\phi(\vec{x})$ is the initial position and $\psi(\vec{x})$ is the initial velocity. It is clear from the physics we need to specify both $u(\vec{x}, t_0)$ and $u_t(\vec{x}, t_0)$.

Note for most of the PDEs we have looked at there is a natural domain $D$ where the PDE is valid. For the guitar string it is the interval between the endpoints, if you wanted to heat up a sheet of metal, it would be the region occupied by the metal, for a dye, it would be the container. In each case the region has a boundary, the endpoints of the guitar; the edges of the metal; the surface of the container.

A **boundary condition** specifies $u$ on the boundary. There are three main types of boundary conditions:

(D) $u$ is specified ("Dirichlet condition").
(N) the outward derivative $\frac{\partial u}{\partial n}$ is specified ("Neumann condition").
(R) $\frac{\partial u}{\partial n} + an$ is specified ("Robin condition").

where $a$ is a function of $x$, $y$ and $z$.

If the boundary condition is simply that $u$ (or $\frac{\partial u}{\partial n}$) is zero then we call the boundary condition **homogeneous**; otherwise we call the boundary condition **inhomogeneous**.

In one dimension, the region is an interval, say $0 < x < l$ and the boundary is just the two endpoints $x = 0$ and $x = l$;

(D) $u(0, t) = g(t)$ and $u(l, t) = h(t)$
(N) $u_x(0, t) = g(t)$ and $u_x(l, t) = h(t)$.

For the vibrating string, we have Dirichlet boundary conditions, $u(0, t) = 0$ and $u(l, t) = 0$. On the other hand, if one end of the string is free to move up and down (say on a frictionless track) then we have the Neumann boundary condition $u_x(l,t) = 0$.

For diffusion, if the container is impermeable, then we get the Neumann boundary condition

$$\frac{\partial u}{\partial n} = 0$$

on the boundary. If at the other extreme the container is perfectly permeable then the concentration at the boundary is zero

$$u = 0.$$

It is a similar story for heat. If the container is perfectly insulated then we get the Neumann boundary condition

$$\frac{\partial u}{\partial n} = 0$$

on the boundary. If at the other extreme the container is surrounded by a heat sink (a large quantity of liquid) then the concentration at the boundary is zero

$$u = g(t)$$

where $g(t)$ is the temperature of the liquid at time $t$.

Sometimes the domain is infinite, for example for the hydrogen atom. In this case we sometimes impose a condition at infinity. For the hydrogen atom we require an integral over space to be one:

$$\int_{\mathbb{R}^3} |\vec{u}| \, \mathrm{d}\vec{x} = 1.$$

(For those people who know their quantum mechanics, the integral computes a probability).

Sometimes one might divide the region into parts and specify different conditions on each part (for example, for a coffee cup, there is the water and the cup itself).