3. Flows vibrations and diffusions

In this section we give some quick derivations of some simple PDEs that arise in physics.

Example 3.1 (Simple transport)**.** *Consider a fluid, water say, flowing at a constant rate c in a horizontal pipe of fixed cross-section in the positive x direction.*

A substance, a pollutant say, is suspended in the water. Let $u(x,t)$ be its concentration in grams/centimeter at time t. Then

$$u_t + cu_x = 0,$$

that is, the rate of change of concentration is proportional to the gradient (we assume that diffusion is negligible).

Solving this constant coefficient linear PDE, we see that the solution is a function of $x - ct$. The means that the substance moves right at a constant speed c.

We give a quick derivation of this PDE. The amount of pollutant in the interval $[0, b]$ at the time t in grams is

$$M = \int_0^b u(x,t)\,\mathrm{d}x.$$

At the time $t + h$ the same molecules of pollutant have all moved to the right by $c \cdot h$ centimeters. It follows that

$$M = \int_0^b u(x,t)\,\mathrm{d}x = \int_{ch}^{b+ch} u(x,t+h)\,\mathrm{d}x.$$

If we differentiate with respect to b we get

$$u(b,t) = u(b+ch, t+h).$$

If we differentiate with respect to h and put $h = 0$ we get

$$0 = cu_x(b,t) + u_t(b,t),$$

which is the equation in (3.1).

Example 3.2 (Vibrating string)**.** *Imagine a flexible, elastic homogeneous string of length l, which undergoes small transverse vibrations, for example, a guitar string.*

We assume that the string remains in a plane. We choose coordinates so that the ends of the string are at the origin and at the point $(l, 0)$. Let $u(x,t)$ denote its vertical position, the displacement from its equilibrium position, at time t and position x.

As the string is perfectly flexible, the tension (or force) is directed tangentially along the string. Let $T(x,t)$ be the magnitude of the tension vector. Let ρ be the density (mass per unit length) of the string, a constant as the string is homogeneous.

Recall the vector form of Newton's law:

$$\vec{F} = m\vec{a}.$$

By assumption

$$|\vec{F}| = T(x,t).$$

We separate the motion into two parts, up and down and side to side. The slope of the string at x_2 is $u_x(x_2,t)$. If we draw a triangle with sides 1 and u_x the other side has length

$$\sqrt{1+u_x^2}.$$

The force then has components

$$\frac{T}{\sqrt{1+u_x^2}} \qquad \text{and} \qquad \frac{Tu_x}{\sqrt{1+u_x^2}}.$$

We assume that the string only moves up and down. We consider what happens over a piece of the string from x_0 to x_1. We assume that the string is roughly straight over this piece so that the force is just the difference of the two forces at the ends x_0 and x_1. We integrate both sides of Newton's law over the string:

$$\begin{aligned}\left[\frac{T}{\sqrt{1+u_x^2}}\right]_{x_0}^{x_1} &= 0\\ \left[\frac{Tu_x}{\sqrt{1+u_x^2}}\right]_{x_0}^{x_1} &= \int_{x_0}^{x_1} \rho u_{tt}\,\mathrm{d}x.\end{aligned}$$

The first equation represents what happens horizontally and the second vertically.

If we assume that u_x is small then we can replace

$$\sqrt{1+u_x^2}$$

by 1. Indeed, if we apply the binomial (or Taylor's) theorem we get

$$(1+u_x^2)^{1/2} = 1 + \frac{1}{2}u_x^2 + \dots.$$

As we are assuming that u_x is small, it follows that u_x^2 is even smaller and so we can ignore u_x^2 and the higher terms (it is easy to justify this approximation, for example by using Taylor's theorem with remainder). The first equation then implies that T is constant along the string. We assume that T is independent of t as well.

If we differentiate the second equation then we get

$$(Tu_x)_x = \rho u_{tt}.$$

Putting all of this together we get the **wave equation:**

$$u_{tt} = c^2 u_{xx} \qquad \text{where} \qquad c = \sqrt{\frac{T}{\rho}}.$$

It will turn out that c is the **wave speed**.

There are some variations on a theme:

(i) If we want to account of air resistance then we get an extra term proportional to the speed:

$$u_{tt} - c^2 u_{xx} + r u_t = 0.$$

(ii) If there is a transverse elastic force (such as a coiled spring) then there is an extra term proportional to the displacement:

$$u_{tt} - c^2 u_{xx} + k u = 0.$$

(iii) If there is an externally applied force, there is an extra term

$$u_{tt} - c^2 u_{xx} = f(x, t),$$

and the equation becomes inhomogeneous.

Example 3.3 (Vibrating drum)**.** *The two dimensional version of string is an elastic, flexible, homogeneous drumhead, that is, a membrane stretched over a frame.*

Suppose that the frame lies in the (x, y)-plane and let $u(x, y, t)$ be the vertical displacement. Following an argument similar to the one for the vibrating string (but which uses vector calculus, see the book for the derivation), we get the two dimensional wave equation

$$u_{tt} = c^2 (u_{xx} + u_{yy}) \qquad \text{where} \qquad c = \sqrt{\frac{T}{\rho}}.$$

The operator

$$\mathscr{L} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

is known as the two dimensional Laplacian. It is denoted Δ or ∇^2.

The three dimensional wave equation takes the same form but where the RHS has the extra term u_{zz}.

Example 3.4 (Diffusion)**.** *Imagine a motionless liquid filling a straight tube and a chemical substance, a dye say, diffusing through the liquid.*

The dye moves from regions of higher concentration to regions of lower concentration. The rate of motion is proportional to the concentration gradient (known as Fick's law of diffusion). Let $u(x,t)$ be the concentration (mass be unit length) of the dye at position x of the pipe at time t.

The mass at time t in the section of pipe from x_0 to x_1 is

$$M(t) = \int_{x_0}^{x_1} u(x,t)\,\mathrm{d}x,$$

so that differentiating under the integral sign, we get

$$\frac{\mathrm{d}M(t)}{\mathrm{d}t} = \int_{x_0}^{x_1} u_t(x,t)\,\mathrm{d}x,$$

The mass only changes by entering or leaving at the endpoints and so by Fick's law

$$\begin{aligned}\frac{\mathrm{d}M(t)}{\mathrm{d}t} &= \text{flow in} - \text{flow out}\\ &= ku_x(x_1,t) - ku_x(x_0,t),\end{aligned}$$

for some constant k. Equating the two expressions for the LHS, we get

$$\int_{x_0}^{x_1} u(x,t)\,\mathrm{d}x = ku_x(x_1,t) - ku_x(x_0,t).$$

If we differentiate with respect to x we get the **diffusion equation**

$$u_t = ku_{xx}.$$

In three dimensions we get

$$u_t = k(u_{xx} + u_{yy} + u_{zz}) = k\Delta u.$$

We define an operator

$$\nabla = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k},$$

as follows. If we have a scalar function, $f(x,y,z)$,

$$f\colon \mathbb{R}^3 \longrightarrow \mathbb{R}$$

then

$$\nabla f = (f_x, f_y, f_z),$$

a function

$$\nabla f\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3.$$

If we have a function

$g\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ given by $g(x,y,z) = (a(x,y,z), b(x,y,z), c(x,y,z))$

then

$$\nabla \cdot g = a_x + b_y + c_z.$$

Note that

$$\nabla^2 f = \nabla \cdot (\nabla f) = \Delta f,$$

the Laplacian of f.

If there is an external source of dye and the rate of diffusion is variable then the diffusion equation becomes

$$u_t = \nabla \cdot (k \nabla f) + f(x,t).$$

Example 3.5 (Heat flow)**.** *Let $u(x,y,z,t)$ be the temperature and let $H(t)$ be the heat (in calories say) in a region D of space.*

The **heat equation** is the PDE

$$c\rho u_t = \nabla \cdot (\kappa \nabla f).$$

Here c is the specific heat of the material (the amount of energy it takes to raise the material by a set temperature), ρ is its density and κ is the heat conductivity.

Note that this has the same form as the diffusion equation.

In all four of the previous examples, suppose that we are in a physical state where the situation is constant in time, so that $u_t = u_{tt} = 0$. For both the wave and the heat equation, the PDE reduces to

$$\Delta u = 0,$$

Laplace's equation. Solutions to Laplace's equation are called **harmonic functions**.

Example 3.6 (Hydrogen atom)**.** *Consider an electron moving around a proton.*

Let m be the mass of the electron, e its change, and $\hbar$ Planck's constant divided by 2π. Put the proton at the origin of coordinates and let

$$r = (x^2 + y^2 + z^2)^{1/2}$$

be the distance to the origin. Then the motion of the electron is given by a "wave function" $u(x,y,z,t)$ which satisfies Schrödinger's wave equation

$$-i\hbar u_t = \frac{\hbar^2}{2m}\Delta u + \frac{e^2}{r} u.$$