21. Parity periodicity and complexity

We say that a function ϕ is **periodic**, with **period** p, if

$$\phi(x+p) = \phi(x) \qquad \text{for all} \qquad x.$$

If ϕ is periodic, with period p, then ϕ is periodic with period np for any positive integer n. If $\phi(x)$ has period p then $\phi(ax)$ has period p/a.

Note that if ϕ_1 and ϕ_2 are periodic with period p then $a\phi_1$ also has period p and so does ϕ_1 and ϕ_2.

Finally, given a function ϕ defined on an interval of length p there is unique extension of ϕ to the whole real line of a function with period p (well, strictly speaking the function is not defined at the two endpoints of the interval).

As cos and sin have period 2π it follows that $\cos \pi x/l$ and $\sin \pi x/l$ have period $2l$ and so $\cos n\pi x/l$ and $\sin n\pi x/l$ also have period $2l$. But then the full Fourier series has period $2l$. Given a function ϕ on $(-l, l)$ there is a unique function on the whole real line with period $2l$, and this is equal to the Fourier series.

Now if we start with a function ϕ on $(0, l)$ there is a unique extension of ϕ to an odd function on the interval $(-l, l)$ and a unique extension of this to a periodic function on the whole real line. This is the same as the Fourier sine series.

On the other hand, if we start with a function ϕ on $(0, l)$ there is a unique extension of ϕ to an even function on the interval $(-l, l)$ and a unique extension of this to a periodic function on the whole real line. This is the same as the Fourier cosine series.

Put differently, the Fourier sine series on $(0, l)$ is the same as a Fourier series on $(-l, l)$ of an odd function. Similarly the Fourier cosine series on $(0, l)$ is the same as a Fourier series on $(-l, l)$ of an odd function.

We can match this to boundary conditions. If we want to solve the wave/diffusion equation on $(0, l)$ with Dirichlet boundary conditions then we want the Fourier sine series, since we want an even function.

If we want to solve the wave/diffusion equation on $(0, l)$ with Neumann boundary conditions then we want the Fourier cosine series, since we want an even function.

If we want to solve the wave/diffusion equation on $(-l, l)$ with periodic boundary conditions then we want the Fourier series.

It is sometimes convenient to work with complex functions, rather than real functions, when we are looking at eigenfunctions of

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2}$$

on $(-l, l)$. Recall that DeMoivre's theorem,

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

It follows that

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \text{and} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

In fact

$$e^{in\pi x/l} \qquad \text{and} \qquad e^{-in\pi x/l}$$

are two eigenfunctions with eigenvalue

$$\lambda = \left(\frac{n\pi}{l}\right)^2.$$

In fact instead of writing $\phi(x)$ in terms of cosine and sine, instead we can write as a sum of exponentials, with complex coefficients:

$$\phi(x) = \sum_{n=-\infty}^{n=\infty} c_n e^{in\pi x/l}.$$

Note that this is a double sum, the index goes from $-\infty$ to ∞.

It is interesting to note that the same magic formula for integration is still valid:

$$\begin{aligned}
\int_{-l}^{l} e^{in\pi x/l} e^{-im\pi x/l} \, \mathrm{d}x &= \int_{-l}^{l} e^{i(n-m)\pi x/l} \, \mathrm{d}x \\
&= \left[\frac{l}{i(n-m)\pi} e^{i(n-m)\pi x/l}\right]_{-l}^{l} \\
&= \frac{l}{i(n-m)\pi}((-1)^{n-m} - (-1)^{m-n}) \\
&= 0,
\end{aligned}$$

provided $n \neq m$. If $n = m$ then we are integrating 1 over $(-l, l)$ and the answer is $2l$.

As before, this implies there is a simple formula for the coefficients c_m:

$$c_m = \frac{1}{2l} \int_{-l}^{l} \phi(x) e^{-im\pi x/l} \, \mathrm{d}x.$$