20. Fourier series

We now turn to the problem of writing a function as an infinite sum of trigonometric functions.

We start with the **Fourier sine series**

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}.$$

We suppose that ϕ is defined on the interval$(0, l)$. We would like to determine the coefficients $A_1, A_2, \dots,$ starting with ϕ. We assume that the RHS converges to ϕ on $(0, l)$.

This turns out to be easy, using the following simple:

Lemma 20.1. *If m and n are positive integers then*

$$\int_0^l \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l} \, \mathrm{d}x = \begin{cases} 0 & \text{if } m \neq n \\ \frac{l}{2} & \text{if } m = n. \end{cases}$$

Proof. We start with the identity

$$\sin \alpha \sin \beta = \frac{1}{2} \cos(\alpha - \beta) - \frac{1}{2} \cos(\alpha + \beta),$$

which is easily checked by applying the addition formula for cosine.

If we assume that $m \neq n$, it follows that

$$\begin{aligned} \int_0^l \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l} \, \mathrm{d}x &= \frac{1}{2} \int_0^l \cos \frac{(n-m)\pi x}{l} - \cos \frac{(n+m)\pi x}{l} \, \mathrm{d}x \\ &= \frac{1}{2\pi} \left[\frac{1}{n-m} \sin \frac{(n-m)\pi x}{l} - \frac{1}{n+m} \sin \frac{(n+m)\pi x}{l} \right]_0^l \\ &= 0. \end{aligned}$$

The last line follows, as we are evaluating sine at four different integer multiples of π, where it is zero.

Now if $m = n$ the calculation changes only in the first term:

$$\cos \frac{(n-m)\pi x}{l} = 1.$$

If we integrate this from 0 to l we get l. □

We are assuming that

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}$$

where both sides are defined on the interval $(0, l)$ and the sum converges there.

If we multiply both sides by

$$\sin \frac{m\pi x}{l}$$

then we get

$$\phi(x) \sin \frac{m\pi x}{l} = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l}.$$

Now integrate both sides from 0 to l:

$$\begin{aligned}
\int_0^l \phi(x) \sin \frac{m\pi x}{l} \, \mathrm{d}x &= \int_0^l \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l} \, \mathrm{d}x \\
&= \sum_{n=1}^{\infty} A_n \int_0^l \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l} \, \mathrm{d}x \\
&= A_m \frac{l}{2}.
\end{aligned}$$

Thus

$$A_m = \frac{2}{l} \int_0^l \phi(x) \sin \frac{m\pi x}{l} \, \mathrm{d}x.$$

Note that the key step is switching the order of integration and summation to get from the first line to the second line.

Example 20.2. *Find the Fourier sine series for $\phi(x) = 1$.*

We just need to compute

$$\begin{aligned}
A_m &= \frac{2}{l} \int_0^l \phi(x) \sin \frac{m\pi x}{l} \, \mathrm{d}x \\
&= \frac{2}{l} \int_0^l \sin \frac{m\pi x}{l} \, \mathrm{d}x \\
&= \frac{2}{l} \left[\frac{-l}{m\pi} \cos \frac{m\pi x}{l} \right]_0^l \\
&= \frac{2(1 - (-1)^m)}{m\pi}.
\end{aligned}$$

It follows that

$$A_m = \begin{cases} \frac{4}{m\pi} & \text{if } m \text{ is odd} \\ 0 & \text{if } m \text{ is even.} \end{cases}$$

Hence

$$1 = \frac{4}{\pi} \left(\sin \frac{\pi x}{l} + \frac{1}{3} \sin \frac{3\pi x}{l} + \frac{1}{5} \sin \frac{5\pi x}{l} + \dots \right).$$

Example 20.3. *Find the Fourier sine series for $\phi(x) = x$.*

We just need to compute

$$\begin{aligned} A_m &= \frac{2}{l}\int_0^l \phi(x)\sin\frac{m\pi x}{l}\,\mathrm{d}x \\ &= \frac{2}{l}\int_0^l x\sin\frac{m\pi x}{l}\,\mathrm{d}x \\ &= \left[\frac{-2x}{m\pi}\cos\frac{m\pi x}{l} + \frac{2l}{m^2\pi^2}\sin\frac{m\pi x}{l}\right]_0^l \\ &= (-1)^{m+1}\frac{2l}{m\pi}. \end{aligned}$$

Hence

$$x = \frac{2l}{\pi}\left(\sin\frac{\pi x}{l} - \frac{1}{2}\sin\frac{2\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + \dots\right).$$

Example 20.4. *Solve the PDE*

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \\ u(0,t) &= u(l,t) = 0 \\ u(x,0) &= x \qquad u_t(x,0) = 0. \end{aligned}$$

The general solution of the PDE, subject to the boundary conditions is

$$u(x,t) = \sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi ct}{l} + B_n\sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l}.$$

The fact that $u_t(x,0)$ implies that $B_n = 0$ for all n. We want to choose A_m so that

$$x = \sum_{n=1}^{\infty} A_n\sin\frac{n\pi x}{l}.$$

We solved this in (20.3). Using this, we get

$$u(x,t) = \frac{2l}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l}.$$

We now look at some variations on a theme. The first is to consider the **Fourier cosine series**

$$\phi(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n\cos\frac{n\pi x}{l}.$$

Once again we assume that both sides are defined on the interval $(0,l)$ and once again we assume that the series converges.

The key result is again a simple result about integration

Lemma 20.5. *If m and n are non-negative integers then*

$$\int_0^l \cos\frac{n\pi x}{l}\cos\frac{m\pi x}{l}\,\mathrm{d}x = \begin{cases} 0 & \text{if } m \neq n \\ \frac{l}{2} & \text{if } m = n > 0. \\ l & \text{if } m = n = 0. \end{cases}$$

As before we multiply both sides by

$$\cos\frac{m\pi x}{l}$$

and integrate over the interval $(0, l)$.

$$\begin{aligned} \int_0^l \phi(x)\cos\frac{m\pi x}{l}\,\mathrm{d}x &= \int_0^l \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l}\cos\frac{m\pi x}{l}\,\mathrm{d}x \\ &= \sum_{n=1}^{\infty} A_n \int_0^l \cos\frac{n\pi x}{l}\cos\frac{m\pi x}{l}\,\mathrm{d}x \\ &= A_m\frac{l}{2}. \end{aligned}$$

Thus

$$A_m = \frac{2}{l}\int_0^l \phi(x)\cos\frac{m\pi x}{l}\,\mathrm{d}x.$$

Note that this finally explains the unusual factor of $1/2$ in front of $A_0/2$.

Example 20.6. *Find the Fourier cosine series for $\phi(x) = 1$.*

We just need to compute. If $m > 0$ then

$$\begin{aligned} A_m &= \frac{2}{l}\int_0^l \phi(x)\cos\frac{m\pi x}{l}\,\mathrm{d}x \\ &= \frac{2}{l}\int_0^l \cos\frac{m\pi x}{l}\,\mathrm{d}x \\ &= \frac{2}{l}\left[\frac{l}{m\pi}\sin\frac{m\pi x}{l}\right]_0^l \\ &= 0. \end{aligned}$$

If $m = 1$ then in fact we get $A_0 = 2$.

It follows that

$$A_m = \begin{cases} 1 & \text{if } m = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$1 = 1.$$

In retrospect we could have guessed this expansion.

Example 20.7. *Find the Fourier cosine series for $\phi(x) = x$.*

We just need to compute

$$\begin{aligned}
A_m &= \frac{2}{l}\int_0^l \phi(x)\cos\frac{m\pi x}{l}\,\mathrm{d}x \\
&= \frac{2}{l}\int_0^l x\cos\frac{m\pi x}{l}\,\mathrm{d}x \\
&= \left[\frac{2x}{m\pi}\sin\frac{m\pi x}{l} + \frac{2l}{m^2\pi^2}\cos\frac{m\pi x}{l}\right]_0^l \\
&= \frac{2l}{m\pi}((-1)^m - 1).
\end{aligned}$$

It follows that

$$A_m = \begin{cases} \frac{-4l}{m^2\pi^2} & \text{if } m \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$x = \frac{l}{2} - \frac{4l}{\pi^2}\left(\cos\frac{\pi x}{l} + \frac{1}{9}\cos\frac{3\pi x}{l} + \frac{1}{25}\cos\frac{5\pi x}{l} + \dots\right).$$

The final possibility is to consider the **full Fourier series** on the interval $(-l, l)$.

$$\phi(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi x}{l} + B_n\sin\frac{n\pi x}{l}\right).$$

Once again there are similar identities

$$\begin{aligned}
&\int_{-l}^{l}\cos\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\,\mathrm{d}x = 0 \\
&\int_{-l}^{l}\cos\frac{n\pi x}{l}\cos\frac{m\pi x}{l}\,\mathrm{d}x = 0 \qquad \text{provided} \qquad m \neq n \\
&\int_{-l}^{l}\sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\,\mathrm{d}x = 0 \qquad \text{provided} \qquad m \neq n.
\end{aligned}$$

We also have

$$\int_{-l}^{l} \cos^2 \frac{n\pi x}{l}\,\mathrm{d}x = l$$
$$\int_{-l}^{l} \sin^2 \frac{n\pi x}{l}\,\mathrm{d}x = l$$
$$\int_{-l}^{l} 1^2\,\mathrm{d}x = 2l.$$

It follows that

$$A_m = \frac{1}{l}\int_{-l}^{l} \phi(x)\cos\frac{m\pi x}{l}\,\mathrm{d}x$$
$$B_m = \frac{1}{l}\int_{-l}^{l} \phi(x)\sin\frac{m\pi x}{l}\,\mathrm{d}x.$$

Example 20.8. *Find the Fourier series for* $\phi(x) = x$.

We just need to compute

$$\begin{aligned}
A_m &= \frac{1}{l}\int_{-l}^{l} \phi(x)\cos\frac{m\pi x}{l}\,\mathrm{d}x \\
&= \frac{1}{l}\int_{-l}^{l} x\cos\frac{m\pi x}{l}\,\mathrm{d}x \\
&= \left[\frac{x}{m\pi}\sin\frac{m\pi x}{l} + \frac{l}{m^2\pi^2}\cos\frac{m\pi x}{l}\right]_{-l}^{l} \\
&= 0.
\end{aligned}$$

On the other hand

$$\begin{aligned}
B_m &= \frac{1}{l}\int_{-l}^{l} \phi(x)\sin\frac{m\pi x}{l}\,\mathrm{d}x \\
&= \frac{1}{l}\int_{-l}^{l} x\sin\frac{m\pi x}{l}\,\mathrm{d}x \\
&= \left[\frac{-x}{m\pi}\cos\frac{m\pi x}{l} + \frac{l}{m^2\pi^2}\sin\frac{m\pi x}{l}\right]_{-l}^{l} \\
&= (-1)^{m+1}\frac{2l}{m\pi}.
\end{aligned}$$

This gives us exactly the same series

$$x = \frac{2l}{\pi}\left(\sin\frac{\pi x}{l} - \frac{1}{2}\sin\frac{2\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + \dots\right).$$

as in (20.3).

In fact we could have guessed this, using the fact that $\phi(x) = x$ is an odd function.