2. First order linear

In this section we will see how to solve first order linear partial differential equations for a function of two variables.

As a warmup, we start with the constant coefficient case. The general constant coefficient first order partial differential equation takes the form

$$au_x + bu_y = 0,$$

where a and b are constants.

We look at two ways to figure out the solution to this PDE. The first is geometric and the second more algebraic, but they are also clearly two sides of the same coin.

First note that if we start with the very simple example of

$$u_x = 0,$$

then the general solution is

$$u(x, y) = f(y),$$

where $f(y)$ is an arbitrary function of y. The solution $u(x, y)$ is constant on the horizontal line $y = c$, c a constant.

Geometric method:

Consider the vector

$$\vec{v} = a\hat{\imath} + b\hat{\jmath} = (a, b).$$

As

$$\begin{aligned} \nabla_{\vec{v}} u &= \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} \right) u \cdot (a\hat{\imath} + b\hat{\jmath}) \\ &= \left(\frac{\partial u}{\partial x}\hat{\imath} + \frac{\partial u}{\partial y}\hat{\jmath} \right) \cdot (a\hat{\imath} + b\hat{\jmath}) \\ &= au_x + bu_y, \end{aligned}$$

is the directional derivative of u in the direction of $\vec{v}$, the equation

$$au_x + bu_y = 0,$$

is equivalent to requiring that the directional derivative is zero. But then u is constant along the lines parallel to $\vec{v}$.

The vector

$$(-b, a)$$

is orthogonal to $\vec{v}$ so that the lines

$$bx - ay = c$$

where c is a constant, are parallel to $\vec{v}$. The solution $u(x, y)$ is constant on every such line, so that $u(x, y)$ only depends on the quantity $bx - ay$. It follows that the general solution to the PDE is

$$u(x, y) = f(bx - ay),$$

where f is any function of one variable. In fact, given c, the function $u(x, y)$ has only one value $f(c)$ and as $c = bx - ay$ we get

$$u(x, y) = f(c) = f(bx - ay).$$

Algebraic method:

The idea is to change variables (or better, make a change of coordinates):

$$x' = ax + by \qquad \text{and} \qquad y' = bx - ay.$$

We have to express the PDE using the new coordinates. To do this, we use the chain rule:

$$\begin{aligned} u_x &= \frac{\partial u}{\partial x} \\ &= \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial x} \\ &= au_{x'} + bu_{y'}. \end{aligned}$$

Similarly

$$\begin{aligned} u_y &= \frac{\partial u}{\partial y} \\ &= \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial y} \\ &= bu_{x'} - au_{y'}. \end{aligned}$$

It follows that

$$\begin{aligned} au_x + bu_y &= a(au_{x'} + bu_{y'}) + b(bu_{x'} - au_{y'}) \\ &= (a^2 + b^2)u_{x'}. \end{aligned}$$

As $a^2 + b^2 \neq 0$, it follows that the old equation

$$au_x + bu_y = 0,$$

reduces to

$$u_{x'} = 0.$$

The general solution to this PDE is

$$u(x', y') = f(y')$$

and as $y' = bx - ay$ this reduces to the same solution

$$u(x, y) = f(bx - ay),$$

we get using the geometric method.

Example 2.1. *Solve*

$$12u_x - 5u_y = 0,$$

including the auxiliary condition that $u(0, y) = y^2$.

We know that the general solution is

$$u(x, y) = f(-5x - 12y).$$

If we put $x = 0$ this reduces to

$$y^2 = u(0, y) = f(-12y).$$

If we put $w = -12y$ then

$$f(w) = \frac{w^2}{144}.$$

Thus the solution is

$$u(x, y) = \frac{(5x + 12y)^2}{144}.$$

Note that it is easy to check that this is a solution. Indeed

$$u_x = \frac{10(5x + 12y)}{144} \qquad \text{and} \qquad u_y = \frac{24(5x + 12y)}{144}.$$

In this case

$$\begin{aligned} 12u_x - 5u_y &= \frac{10(5x + 12y)}{12} - \frac{10(5x + 12y)}{12} \\ &= 0. \end{aligned}$$

On the other hand

$$\begin{aligned} u(0, y) &= \frac{(12y)^2}{144} \\ &= y^2, \end{aligned}$$

as required.

One can push this method a little bit further to the case of variable coefficients.

Example 2.2. *Solve*

$$u_x + yu_y = 0.$$

This is a linear PDE with variable coefficient y in front of u_y. At the point (x, y) the solution has directional derivative zero in the direction of $(1, y)$. The curves in the (x, y)-plane with tangent vectors $(1, y)$ have slopes y. Hence their equations are

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{1}.$$

The general solution of this ODE is

$$y = Ce^x.$$

Note that there is one curve for each value of C and these curves cover the (x, y)-plane, once, and once only. These curves are called the **characteristic curves** of the PDE.

$u(x, y)$ is constant on these curves as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x} u(x, Ce^x) &= \frac{\partial u}{\partial x} + Ce^x \frac{\partial u}{\partial y} \\ &= u_x + y u_y \\ &= 0.\end{aligned}$$

It follows that $u(x, y)$ is a function only of

$$C = ye^{-x}.$$

Thus

$$u(x, y) = f(e^{-x} y)$$

is the general solution of this PDE, where f is an arbitrary function of one variable.

Example 2.3. *Solve the PDE in* (2.2) *subject to the condition*

$$u(0, y) = y^2.$$

We want

$$\begin{aligned}y^2 &= u(0, y) \\ &= f(e^{-0} y) \\ &= f(y).\end{aligned}$$

Thus

$$\begin{aligned}u(x, y) &= (e^{-x} y)^2 \\ &= e^{-2x} y^2.\end{aligned}$$

Example 2.4. *Solve the PDE*

$$u_x + (3x^2 y) u_y = 0.$$

We first solve the ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 y$$

to find the characteristic curves. By separation of variables we have

$$\log y = x^3 + c.$$

Thus the characteristic curves are given by

$$y = Ce^{x^3}.$$

It follows that

$$u(x, y) = f(ye^{-x^3})$$

is the general solution, where f is an arbitrary function of one variable.

It is easy to check that these are solutions. We have

$$u_x = -3x^2 ye^{-x^3} f'(ye^{-x^3}) \qquad \text{and} \qquad u_y = e^{-x^3} f'(ye^{-x^3}).$$

In this case

$$\begin{aligned} u_x + 3x^2 y u_y &= -3x^2 ye^{-x^3} f'(ye^{-x^3}) + 3x^2 ye^{-x^3} f'(ye^{-x^3}) \\ &= 0. \end{aligned}$$