19. Robin conditions

We consider the possibility that we have Robin boundary conditions, for either the diffusion or the wave equation. After separating variables, this means we are considering

$$X'' = -\lambda X$$

where

$$\begin{aligned} X' - a_0 X &= 0 \qquad \text{at} \qquad x = 0 \\ X' + a_l X &= 0 \qquad \text{at} \qquad x = l. \end{aligned}$$

If we are looking at the temperature in a metal rod then *radiation* of energy at both ends corresponds to a_0 and a_l are both positive, *absorption* of energy at both ends corresponds to a_0 and a_l are both negative, and *insulation* of energy at both ends if a_0 and a_l are both zero.

On the other hand, if we are considering a vibrating string then the string shares its energy with the endpoints if a_0 and a_l are both positive and the string gains energy from the endpoints if a_0 and a_l are both negative.

Note that at the endpoint $x = 0$ moving to the left corresponds to leaving the region $0 < x < l$ and at the endpoint $x = l$ moving to the right corresponds to leaving the region $0 < x < l$. This explains why there is a difference in sign between a_0 and a_l.

We now look for solutions to the ODE. It will turn out that most eigenvalues are positive. In special conditions there might be one negative eigenvalue.

We first look for positive eigenvalues,

$$\lambda = \beta^2 > 0.$$

In this case the solution of the ODE is

$$X(x) = C \cos \beta x + D \sin \beta x.$$

It follows that

$$X'(x) \pm aX(x) = (\beta D \pm aC) \cos \beta x + (-\beta C \pm aD) \sin \beta x$$

If we impose the boundary conditions this gives us a pair of linear equations for C and D.

$$\begin{aligned} \beta D - a_0 C &= 0 \\ (\beta D + a_l C) \cos \beta l + (-\beta C + a_l D) \sin \beta l &= 0 \end{aligned}$$

If we convert these equations to matrix form we get

$$\begin{pmatrix} -a_0 & \beta \\ a_l \cos\beta l - \beta \sin \beta l & \beta \cos\beta l + a_l \sin\beta l \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

If λ is an eigenvalue then this matrix has a non-trivial solution. The condition that this equation has a non-trivial solution is equivalent to the condition that the determinant is zero. Thus we must have

$$a_0(\beta\cos\beta l + a_l \sin\beta l) = \beta(\beta\sin\beta l - a_l\cos\beta l).$$

Dividing through by $\cos\beta l$ and solving for $\tan\beta l$ we get

$$\tan\beta l = \frac{(a_0 + a_l)\beta}{\beta^2 - a_0 a_l}.$$

The corresponding eigenfunction is

$$X(x) = \beta\cos\beta x + a_0 \sin\beta x.$$

It is interesting to note that there is an exceptional case when $\cos\beta l = 0$. It is not hard to see that this means $\beta = \sqrt{a_0 a_l}$.

We now try to solve for β. It is not possible, in general, to find an exact formula for the solutions. One can either try to determine approximations of the numerical values or determine the qualitative behaviour of the solutions. We will do the latter, by considering the graph of both sides of the equation.

We view both sides of the equation as functions of β; where these two functions intersect determines the eigenvalues (namely, square the corresponding value of β).

The LHS is easy, it is the graph of the tangent function. The RHS is a rational function, whose behaviour depends heavily on a_0 and a_1.

Case I: radiation at both ends: $a_0 > 0$ and $a_1 > 0$. The rational function has an asymptote at $\sqrt{a_0 a_1}$.

From the picture, one can see that

$$\left(\frac{n\pi}{l}\right)^2 < \lambda_n < \left(\frac{(n+1)\pi}{l}\right)^2.$$

The other thing which one can see from the picture is that

$$\lim \beta_n - n\frac{\pi}{l} = 0.$$

There are many other cases. The book deals with another one in detail.

Now suppose that the eigenvalue is negative. Let

$$\lambda = -\gamma^2 < 0.$$

Then

$$X(x) = C\cosh\gamma x + D\sinh\gamma x.$$

The same analysis as before leads to

$$\tanh\gamma l = -\frac{(a_0 + a_l)\gamma}{\gamma^2 + a_0 a_l}.$$

If a_0 and $a_1 > 0$ one can see that there are no negative eigenvalues. If $a_0 < 0$ and $a_l > 0$, and $a_0 + a_l > 0$, so that there is more radiation than absorption, the situation is more complicated.

If

$$0 < \frac{-(a_0 + a_l)}{a_0 a_l} < l,$$

then there is a negative eigenvalue.

Let us consider the physical consequences of a negative eigenvalue. We have

$$u(x,t) = \sum T_n(t) X_n(x)$$

then

$$T_n(t) = \begin{cases} A_n e^{-\lambda_n k t} & \text{for diffusion} \\ A_n \cos\sqrt{\lambda_n} ct + B_n \sin\sqrt{\lambda_n} ct & \text{for waves.} \end{cases}$$

For example, for diffusion, if $\lambda_n < 0$ then we have one term that increases with time.