18. Neumann condition

We will use the method of separation of variables to deal with Neumann boundary conditions on a finite interval. We replace the Dirichlet boundary conditions

$$u(0,t) = u(l,t) = 0$$

with the boundary conditions

$$u_x(0,t) = u_x(l,t) = 0.$$

For the time being we don't worry if we have the diffusion equation or the wave equation.

We look for solutions of the form

$$u(x,t) = X(x)T(t).$$

This gives us the same ODE's as before, except that the boundary conditions have changed for $X(x)$.

$$X'' = -\lambda X \qquad \text{where} \qquad X'(0) = X'(l) = 0.$$

We proceed as before. We first search for the positive eigenvalues $\lambda = \beta^2 > 0$. The general solution of the ODE is

$$X(x) = C\cos\beta x + D\sin\beta x.$$

It follows that

$$X'(x) = -C\beta\sin\beta x + D\beta\cos\beta x.$$

The boundary conditions imply that

$$D\beta = D\beta\cos 0 = X(0) = 0.$$

Thus $D = 0$. But then

$$-C\beta\sin\beta l = 0.$$

As $C \neq 0$, it follows that

$$\beta = \frac{\pi}{l}, \quad \frac{2\pi}{l}, \quad \frac{23\pi}{l}, \quad \dots$$

as before. Thus

$$X_n(x) = \cos\frac{n\pi x}{l}$$

is an eigenfunction with eigenvalue

$$\lambda_n = \left(\frac{n\pi x}{l}\right)^2.$$

One new feature is that 0 is also an eigenvalue. We want to solve

$$X'' = 0.$$

The general solution is

$$X(x) = Ax + B.$$

We have

$$X'(x) = A.$$

The boundary conditions imply that $A = 0$, twice, as it were. So $X(x) = 1$ is an eigenfunction with eigenvalue 0.

It is not hard to rule out negative eigenvalues or even complex eigenvalues.

Suppose we start with the diffusion equation

$$\begin{aligned} u_t &= ku_{xx} && \text{for} && 0 < x < l \\ u(x,0) &= \phi(x) && \text{for} && t = 0 \\ u(0,t) = u(l,t) &= 0 && \text{for} && x = 0, l. \end{aligned}$$

Then we get solutions of the form

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-n^2\pi^2 t/l^2} \cos\frac{n\pi x}{l}.$$

If we plug in $t = 0$ we get the initial conditions

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l}.$$

We will see later the reason to treat the first term differently.

Note that as $t \to \infty$ we each individual term decays very rapidly.

Now consider the wave equation

$$\begin{aligned} u_{tt} &= c^2 u_{xx} && \text{for} && 0 < x < l \\ u(x,0) = \phi(x) & \quad u_t(x,0) = \psi(x) && \text{for} && t = 0 \\ u_x(0,t) = u_x(l,t) &= 0 && \text{for} && x = 0, l. \end{aligned}$$

The twist here is what happens for the eigenvalue $\lambda = 0$. We get the ODE

$$T'' = 0,$$

so that

$$T(t) = A + Bt.$$

Thus we get solutions of the form

$$u(x,t) = \frac{1}{2}A_0 + \frac{1}{2}B_0 t + \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi ct}{l} + B_n \sin\frac{n\pi ct}{l}\right)\cos\frac{n\pi x}{l}.$$

If we plug in $t = 0$ we get the initial conditions

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l},$$

and

$$\psi(x) = \frac{1}{2}B_0 + \sum_{n=1}^{\infty} B_n \frac{n\pi c}{l} \cos\frac{n\pi x}{l}.$$