17. Separation of variables I

We now introduce a simple but powerful technique to solve PDEs. Consider the Dirichlet boundary problem for a finite interval

$$
\begin{aligned}
u_{tt} &= c^2 u_{xx} \qquad \text{for} \qquad 0 < x < l \\
u(x,0) &= \phi(x) \quad u_t(x,0) = \psi(x) \qquad \text{for} \qquad t = 0 \\
u(0,t) &= u(l,t) = 0 \qquad \text{for} \qquad x = 0, l.
\end{aligned}
$$

The idea is to first suppose that the solution has a particularly simple form. It won't be the case that every solution has the same form but it will be the case we can write every solution as a sum of these solutions.

We look for *separated solutions*, that is, solutions of the form

$$u(x,t) = X(x)T(t).$$

Note that t and x are the input variables and $X(x)$ and $T(t)$ are the outputs. For the time being we don't worry about the initial conditions.

If we plug in the separated solution into the wave equation we get

$$X(x)T''(t) = c^2 X''(x)T(t).$$

Dividing by $-c^2X(x)T(t)$ this gives

$$-\frac{T''}{c^2T} = -\frac{X''}{X}.$$

Note that both sides of the equation are constant. Indeed the LHS only depends on t and the RHS only depends on x so that both sides are independent of x and t. Suppose that the constant is λ, so that

$$-\frac{T''}{c^2T} = -\frac{X''}{X} = \lambda.$$

Later we will see that $\lambda > 0$, which is why we use the minus sign.

It follows that $\lambda = \beta^2$ for some $\beta > 0$. Then we get a pair of ODE's,

$$X'' + \beta^2 X = 0 \qquad \text{and} \qquad T'' + \beta^2 T = 0.$$

The general solution of these ODE's is

$$
\begin{aligned}
X(x) &= C\cos\beta x + D\sin\beta x \\
T(t) &= A\cos\beta ct + B\sin\beta ct.
\end{aligned}
$$

Here A, B, C and D are constants.

Now we consider the boundary conditions. These imply that

$$
\begin{aligned}
0 &= X(0) \\
&= C\cos 0 + D\sin 0 \\
&= C,
\end{aligned}
$$

and

$$\begin{aligned} 0 &= X(l) \\ &= C\cos\beta + D\sin\beta l \\ &= D\sin\beta l. \end{aligned}$$

Since we are not interested in the solution $C = D = 0$ we must have that βl is a root of sin, that is,

$$\beta l = n\pi \qquad \text{so that} \qquad \beta = \frac{n\pi}{l}.$$

In this case

$$X_n(x) = \sin\frac{n\pi x}{l}.$$

Putting all of this together we get infinitely many separated solutions

$$u_n(x,t) = \left(A_n \cos\frac{n\pi ct}{l} + B_n \sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l},$$

one for each natural number $n = 1, 2, 3, \dots$. Here A_n and B_n are arbitrary constants.

By linearity any finite sum is a solution to the wave equation

$$u_n(x,t) = \sum_n \left(A_n \cos\frac{n\pi ct}{l} + B_n \sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l}.$$

Now if put $t = 0$ then we get

$$\phi(x) = \sum_n A_n \sin\frac{n\pi x}{l}$$

and

$$\psi(x) = \sum_n B_n \frac{n\pi c}{l} \sin\frac{n\pi cx}{l}.$$

Thus we can solve anyone of these problems when ϕ and ψ have the given form.

It is sensible to ask if one can extend these finite sums to infinite sums (much like going from Taylor polynomials to Taylor series). It then becomes an interesting question to know which functions are represented by these infinite sums. This was the subject of an intense debate that went on for nearly half a century. The stunning conclusion is that pretty much any function ϕ and ψ can be represented this way.

The numbers

$$\frac{n\pi c}{l}$$

in front of t for $T(t)$ are called the *frequencies*.

If we go back to the violin string then the frequencies are

$$\frac{n\pi\sqrt{T}}{l\sqrt{\rho}}.$$

A similar trick works for diffusion.

$$\begin{aligned} u_t &= k u_{xx} && \text{for} && 0 < x < l \\ u(x,0) &= \phi(x) && \text{for} && t = 0 \\ u(0,t) = u(l,t) &= 0 && \text{for} && x = 0, l. \end{aligned}$$

We look for *separated solutions*, that is, solutions of the form

$$u(x,t) = X(x)T(t).$$

If we plug in the separated solution into the wave equation we get

$$X(x)T'(t) = kX''(x)T(t).$$

Dividing by $kX(x)T(t)$ this gives

$$\frac{T'}{kT} = \frac{X''}{X}.$$

Note that both sides of the equation are constant. Indeed the LHS only depends on t and the RHS only depends on x so that both sides are independent of x and t. Suppose that the constant is $-\lambda$, so that

$$\frac{T'}{c^2T} = \frac{X''}{X} = -\lambda.$$

Later we will see that $\lambda > 0$, which is why we use the minus sign.

It follows that $\lambda = \beta^2$ for some $\beta > 0$. Then we get a pair of ODE's,

$$X'' + \beta^2 X = 0 \qquad \text{and} \qquad T' + \beta^2 kT = 0.$$

The general solution of these ODE's is

$$\begin{aligned} X(x) &= C\cos\beta x + D\sin\beta x \\ T(t) &= Ae^{-\beta^2 kt}. \end{aligned}$$

Here A, C and D are constants.

As before, imposing the boundary conditions, we get that $C = 0$ and

$$\beta l = n\pi \qquad \text{so that} \qquad \beta = \frac{n\pi}{l}.$$

In this case

$$X_n(x) = \sin\frac{n\pi x}{l}.$$

Putting all of this together we get infinitely many separated solutions

$$u_n(x,t) = A_n e^{-\frac{n^2\pi^2}{l^2}kt}\sin\frac{n\pi x}{l},$$

one for each natural number $n = 1, 2, 3, \ldots$. Here A_n is an arbitrary constant.

By linearity any finite sum is a solution to the diffusion equation

$$u_n(x,t) = \sum_n A_n e^{-\frac{n^2\pi^2}{l^2}kt} \sin \frac{n\pi x}{l}.$$

Now if we put $t = 0$ then we get

$$\phi(x) = \sum_n A_n \sin \frac{n\pi x}{l}.$$

Thus we can solve anyone of these problems when ϕ has the given form. Note the factor of each term corresponding to exponential decay.

Imagine the following situation. A cylinder of liquid has some dye in it. Both ends of the liquid open up to a huge vat of liquid. As time progresses, any dye that goes into one of the vats, essentially disappears. Thus as time progresses the amount of dye in the cylinder decays to zero.

The numbers

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2 \qquad \text{and the functions} \qquad X_n = \sin \frac{n\pi x}{l}$$

are sometimes called **eigenvalues** and **eigenfunctions**. In fact they satisfy

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2} X = \lambda X,$$

where $X(0) = X(l) = 0$. In physics and engineering the eigenfunctions are sometimes called the *normal modes*, since they are the solutions which persist for all time.

Finally we check that the eigenvalues are all positive. First why couldn't they be zero? In this case we are solving

$$X'' = 0,$$

so that $X(x) = Ax + B$. The boundary conditions imply that $A = B = 0$. But then $X = 0$ and this is not allowed as an eigenfunction.

What if $\lambda < 0$? Then we get

$$X(x) = A \cosh x + B \sinh x.$$

The boundary conditions still imply that $A = B = 0$.

Finally consider the possibility that λ is a complex number. We can still find β such that $\beta^2 = \lambda$. Then we get

$$X(x) = Ae^{\beta x} + Be^{-\beta x}.$$

The two boundary conditions yield

$$A + B = X(0) = 0 \qquad \text{and} \qquad Ae^{\beta l} + Be^{-\beta l} = X(l) = 0.$$

It follows that $e^{2\beta l} = 1$. This implies that $2\beta l$ is a multiple of $2\pi i$, so that β is purely imaginary. But then $\lambda < 0$.