16. Waves with a source

We solve the inhomogeneous wave equation

$$u_{tt} - c^2 u_{xx} = f(x,t),$$

with the usual initial conditions

$$u(x,0) = \phi(x) \qquad \text{and} \qquad u_t(x,0) = \psi(x).$$

As

$$\mathscr{L} = \partial_t^2 - c^2 \partial_x^2$$

is a linear operator the solution has three terms, one each for f, ϕ and ψ:

$$u(x,t) = \frac{1}{2}\phi(x+ct) + \frac{1}{2}\phi(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi + \frac{1}{2c}\iint_{\Delta} f.$$

Here, Δ is the characteristic triangle. In fact the double integral is

$$\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,\mathrm{d}y\,\mathrm{d}s.$$

The first two terms correspond to ϕ, the third to ψ and the fourth to f. The first three terms come from d'Alembert's formula and the last one is new. As with the heat equation we integrate over both space and time. Not surprisingly, given the principle of causality, we just need to integrate over the characteristic triangle.

We first observe that this problem is well-posed. Recall that this has three parts, existence, uniqueness and stability. Existence follows from the formula; if ϕ is $\mathcal{C}^2$, ψ is $\mathcal{C}^1$ and f is continuous then u is $\mathcal{C}^2$. Uniqueness follows from one of the derivations in the book. Stability just means small changes in ϕ, ψ and f gives small changes in u. This is reasonabbly clear from the formula.

To see that this is the correct formula we use the method of characteristic coordinates. We make the change of variable

$$\xi = x + ct \qquad \text{and} \qquad \eta = x - ct.$$

Recall that in this coordinate system the operator $\mathscr{L}$ takes a particularly simple form.

$$\begin{aligned}\mathscr{L}\,u &= u_{tt} - c^2 u_{xx} \\ &= -4c^2 u_{\xi\eta} \\ &= f\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2c}\right).\end{aligned}$$

If we integrate with respect to η keeping ξ constant then we get

$$-4c^2 u_\xi = \int^\eta f \, \mathrm{d}\eta.$$

Now if we integrate with respect to ξ then we get

$$u = -\frac{1}{4c^2} \int^\xi \int^\eta f \, \mathrm{d}\eta \, \mathrm{d}\xi.$$

Now we have to convert the integral from the characteristic coordinates back to xy-coordinates.

This involves the Jacobian of the change of coordinates,

$$\begin{aligned} J &= \begin{vmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial t} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial t} \end{vmatrix} \\ &= \begin{vmatrix} 1 & c \\ 1 & -c \end{vmatrix} \\ &= 2c. \end{aligned}$$

Recall that areas are all positive, so that even if there is a change in orientation the area is positive. Strictly speaking we should take the absolute value of the determinant of the coordinate change but we suppressed the absolute value to not clutter the displayed algebra.

Finally we have to check that we really do get an integral over the triangle Δ. Clearly we seem to be integrating over a rectangle for ξ and η. When we change coordinates we get the characteristic lines in x and y. Clearly two such lines should be $x - ct$ and $x + ct$ is constant, through the point we fix. But $f(x, t)$ is not defined for $t < 0$ so in fact we also get the condition $t > 0$, so that we get the triangle Δ.