15. Diffusion with a source

Our goal is to solve the inhomogeneous diffusion equation on the whole line

$$u_t - ku_{xx} = f(x,t) \qquad \text{for} \qquad -\infty < x < \infty, \quad 0 < t < \infty,$$

with the initial condition

$$u(x,0) = \phi(x).$$

If $u(x,t)$ is the initial temperature in a rod then $f(x,t)$ represents a heat source and $\phi(x)$ represents the initial temperature.

We claim that the general solution is

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,\mathrm{d}y + \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)\,\mathrm{d}y\,\mathrm{d}s$$

The first term involves only ϕ and is the term we saw before. The second term only involves f and is new. Both integrals involve the source function S.

By linearity, to check this formula, it suffices to check that the second term satisfies the diffusion equation with the source function f. We have

$$\begin{aligned}
u_t(x,t) &= \frac{\partial}{\partial t}\int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)\,\mathrm{d}y\,\mathrm{d}s \\
&= \int_0^t \int_{-\infty}^{\infty} \frac{\partial S}{\partial t}(x-y,t-s)f(y,s)\,\mathrm{d}y\,\mathrm{d}s + \lim_{s\to t}\int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)\,\mathrm{d}y \\
&= \int_0^t \int_{-\infty}^{\infty} k\frac{\partial^2 S}{\partial x^2}(x-y,t-s)f(y,s)\,\mathrm{d}y\,\mathrm{d}s + \lim_{\epsilon\to 0}\int_{-\infty}^{\infty} S(x-y,\epsilon)f(y,s)\,\mathrm{d}y \\
&= k\frac{\partial^2}{\partial x^2}\int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)\,\mathrm{d}y\,\mathrm{d}s + f(x,t) \\
&= k\frac{\partial^2 u}{\partial x^2} + f(x,t).
\end{aligned}$$

To get from the first line to the second line we used the fact that the integral is a function of a function and we differentiated under the integral sign. Note that $S(x,t)$ is not defined when $t = 0$ and so have to take a limit. To get from the second line to the third line we used the fact that S is a solution of the diffusion equation. To get from the third line to the fourth line we pulled the spatial derivative out of the integral sign and we used the fact the second integral is a solution of the diffusion equation with initial condition $f(x,t)$.

Thus $u(x,t)$ is a solution of the inhomogeneous diffusion equation.

Now we check the initial conditions As t goes to zero the first term gives us $\phi(x)$ and the second term is an integral from 0 to 0 and this is zero. Thus $u(x,t)$ is indeed a solution of the inhomogeneous diffusion equation with the given boundary conditions.

If we use the fact that the source function is the Gaussian the second term is

$$\int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi k(t-s)}} e^{-(x-y)^2/4k(t-s)} f(y,s)\,\mathrm{d}y\,\mathrm{d}s.$$

We can use the method of reflection to solve the inhomogeneous diffusion equation on the half-line. In fact we can even solve the inhomogeneous diffusion equation on the half-line with a boundary source.

$$\begin{aligned} v_t - kv_{xx} &= f(x,t) \qquad \text{for} \qquad 0 < x < \infty, \quad 0 < t < \infty \\ v(x,0) &= \phi(x) \qquad \text{for} \qquad t = 0 \\ v(0,t) &= h(t) \qquad \text{for} \qquad x = 0. \end{aligned}$$

In fact we just use the **method of subtraction**, which simply means we consider

$$V(x,t) = v(x,t) - h(t).$$

Then $V(x,t)$ will satisfy

$$\begin{aligned} V_t - kV_{xx} &= f(x,t) - h'(t) \qquad \text{for} \qquad 0 < x < \infty, \quad 0 < t < \infty \\ V(x,0) &= \phi(x) - h(0) \qquad \text{for} \qquad t = 0 \\ v(0,t) &= 0 \qquad \text{for} \qquad x = 0. \end{aligned}$$

We can solve for $V(x,t)$ using the method of reflection and then we get $v(x,t)$ simply by addition.

Note that v is defined in the first quadrant. If $f(0,0) \neq h(0)$ then we get a discontinuity; this makes physical sense (imagine sticking the end of a hot bar into a block of ice).

We can also solve the corresponding Neumann boundary problem:

$$\begin{aligned} w_t - kw_{xx} &= f(x,t) \qquad \text{for} \qquad 0 < x < \infty, \quad 0 < t < \infty \\ w(x,0) &= \phi(x) \qquad \text{for} \qquad t = 0 \\ w_x(0,t) &= h(t) \qquad \text{for} \qquad x = 0. \end{aligned}$$

In this case consider $W(x,t) = w(x,t) - xh(t)$.