14. Reflections of waves

Now we look at the same problem for the wave equation. Consider the Dirichlet boundary problem

$$\begin{aligned} v_{tt} &= c^2 v_{xx} \qquad \text{for} \qquad 0 < x < \infty, \quad 0 < t < \infty \\ v(x,0) = \phi(x) \quad & v_t(x,0) = \psi(x) \qquad \text{for} \qquad t = 0 \\ v(0,t) &= 0 \qquad \text{for} \qquad x = 0. \end{aligned}$$

We employ the same trick as before. Let ϕ_{odd} and ψ_{odd} be the odd extensions of ϕ and ψ to the whole real line. Let $u(x,t)$ be the solution to the wave equation on the whole real line and let v be the restriction of u to positive values of x.

As before, v has the same derivatives as u on the positive real axis, so that it is a solution to the wave equation and $v(0,t) = 0$ as u is odd.

If we apply d'Alembert's formula then we get

$$v(x,t) = \frac{1}{2}\left(\phi_{\text{odd}}(x+ct) + \phi_{\text{odd}}(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{odd}}(y)\,\mathrm{d}y.$$

We now turn this into a formula involving ϕ and ψ. There are two possibilities, depending on the sign of $x - ct$. If $x > c|t|$ then both $x + ct$ and $x - ct$ are positive and so

$$v(x,t) = \frac{1}{2}\left(\phi(x+ct) + \phi(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,\mathrm{d}y,$$

the usual formula. But now suppose that $x < c|t|$. Then $x - ct$ is negative so that

$$\phi_{\text{odd}}(x-ct) = -\phi(ct-x).$$

Thus

$$v(x,t) = \frac{1}{2}\left(\phi(x+ct) - \phi(ct-x)\right) + \frac{1}{2c}\int_{0}^{x+ct} \psi(y)\,\mathrm{d}y + \frac{1}{2c}\int_{x-ct}^{0} -\psi(-y)\,\mathrm{d}y,$$

If we replace y by $-y$ in the second integral then we get

$$v(x,t) = \frac{1}{2}\left(\phi(x+ct) - \phi(ct-x)\right) + \frac{1}{2c}\int_{ct-x}^{x+ct} \psi(y)\,\mathrm{d}y,$$

valid when $x < c|t|$.

We now try to do the same thing for a finite interval. Consider the Dirichlet boundary problem

$$\begin{aligned} v_{tt} &= c^2 v_{xx} \qquad \text{for} \qquad 0 < x < l, \quad 0 < t < \infty \\ v(x,0) = \phi(x) \quad & v_t(x,0) = \psi(x) \qquad \text{for} \qquad t = 0 \\ v(0,t) = v(l,t) &= 0 \qquad \text{for} \qquad x = 0, l. \end{aligned}$$

We employ the same trick as before. Let ϕ_{ext} and ψ_{ext} be the *odd* extensions of ϕ and ψ to the whole real line. More precisely, we want

$$\phi_{\text{ext}}(-x) = -\phi_{\text{ext}}(x) \qquad \text{and} \qquad \phi_{\text{ext}}(2l - x) = -\phi_{\text{ext}}(x).$$

In other words, we want symmetry about the two points $x = 0$ and $x = l$. We let

$$\phi_{\text{ext}}(x) = \begin{cases} \phi(x) & \text{if } 0 < x < l \\ -\phi(-x) & \text{if } -l < x < 0 \end{cases}$$

and we then extend to make ϕ_{ext} periodic with period $2l$, that is,

$$\phi_{\text{ext}}(x + 2l) = \phi_{\text{ext}}(x).$$

Let $u(x,t)$ be the solution to the wave equation on the whole real line with initial conditions ϕ_{ext} and ψ_{ext} and let v be the restriction of u to positive values of x.

As before, v has the same derivatives as u on the positive real axis, so that it is a solution to the wave equation and $v(0,t) = v(l,t) = 0$.

If we apply d'Alembert's formula then we get

$$v(x,t) = \frac{1}{2}\left(\phi_{\text{ext}}(x + ct) + \phi_{\text{ext}}(x - ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{ext}}(y)\,\mathrm{d}y.$$

We now turn this into a formula involving ϕ and ψ. This formula depends heavily on the number of reflections involved. We have two waves, one traveling to the left, one traveling to the right. These waves can reflect off both sides, $x = 0$ and $x = l$.

For example, suppose that $-2l < x - ct < l$ and $2l < x + ct < 3l$, so that the wave traveling left reflects three times and the wave traveling right reflects twice. Every time we reflect there is a change in sign. So the wave traveling left comes with a negative sign and the wave traveling right comes with a positive sign.

We get

$$\phi_{\text{ext}}(x+ct) = -\phi(4l - x - ct) \qquad \text{and} \qquad \phi_{\text{ext}}(x - ct) = \phi(x - ct + 2l).$$

There are similar formulae for ψ_{ext}. We get

$$v(x,t) = \frac{1}{2}\left(\phi(x - ct + 2l) - \phi(4l - x - ct)\right) + \frac{1}{2c}\left(\int_{x-ct}^{-l} \psi(y + 2l)\,\mathrm{d}y + \int_{-l}^{0} -\psi(-y)\,\mathrm{d}y\right.$$
$$\left. + \int_0^l \psi(y)\,\mathrm{d}y + \int_l^{2l} -\psi(-y + 2l)\,\mathrm{d}y + \int_{2l}^{3l} \psi(y - 2l)\,\mathrm{d}y + \int_{3l}^{x+ct} -\psi(-y + 4l)\,\mathrm{d}y\right).$$

If we make the change of variables $y \longrightarrow -y$ and $y - 2l \longrightarrow -y + 2l$ then we see that the second and third integral and the fourth and fifth integrals cancel. Thus the formula reduces to

$$v(x,t) = \frac{1}{2}\left(\phi(x-ct+2l) - \phi(4l-x-ct)\right) + \frac{1}{2c}\int_{x-ct+2l}^{4l-x-ct} \psi(y)\,\mathrm{d}y.$$

There are similar formulae depending on where $x+ct$ and $x-ct$ land in the intervals $[il,(i+1)l]$ of length l.

It is clear that even if these formulae are correct something is missing from this whole picture.