12. Compare and contrast

It is interesting to compare waves and diffusion. Waves spread out at a finite speed and by contrast in diffusion the disturbance smooths out as time progresses.

Property	**Waves**	**Diffusions**
Speed of propogation	Finite ($\leq c$)	Infinite
Singularities for $t > 0$	Transported along characteristics	Disappears immediately
Well-posed for $t > 0$	Yes	Yes (for bounded solutions)
Well-posed for $t < 0$	Yes	No
Maximum principle	No	Yes
Behaviour as $t \to \infty$	Energy is constant	Decays to zero
Information	Transported	Slowly lost.

We have seen most of the properties of the wave function already. It is pretty obvious the maximum principle fails for the wave equation.

We have already seen that the diffusion equation is well-posed for $t > 0$, that there is a maximum principle, that the solution decays to zero and from this it is not hard to see that information is slowly lost. Note that after time $t = 0$ the behaviour at any point depends on what happens at every other point (even if the weight we assign goes to zero); this translates to the fact that the speed of propagation is infinite.

It is interesting to write down an example where well-posedness fails for $t < 0$.

For example, let

$$u_n(x,t) = \frac{1}{n} \sin nx e^{-n^2 kt}.$$

It is easy to see that this satisfies the diffusion equation for all x and t. On the other hand

$$u_n(x,0) = n^{-1} \sin nx \to 0 \qquad \text{uniformly as} \qquad n \to \infty.$$

But consider what happens for $t < 0$. Let us take $t = -1$. Then

$$u_n(x,-1) = n^{-1} \sin nx e^{kn^2} \to \pm\infty \qquad \text{uniformly as} \qquad n \to \infty,$$

unless x is a rational multiple of π.

Thus u_n is close to the zero solution for $t = 0$ but not for $t = -1$. This violates stability.

Of consider $u(x,t) = S(x, t+1)$. This is a solution of the diffusion equation for $t > -1$ and any x. As $t \to -1$ this solution tends to infinity. The initial data is

$$e^{-x^2/4k}.$$

It is also obvious from a physical point of view that one cannot go back in time and figure out what happened in the past from what is happening now.