11. THE DIFFUSION EQUATION ON THE LINE

Our goal is to solve as explicitly as possible the diffusion equation on the whole line

$$u_t = k u_{xx} \qquad \text{for} \qquad -\infty < x < \infty, \quad 0 < t < \infty,$$

with the initial condition

$$u(x, 0) = \phi(x).$$

The method of solution is quite different to previous methods. We start with a particular choice of $\phi(x)$, solve for this and induce all of the other solutions for different choices of $\phi(x)$ from this one solution.

We start with five different ways to go from one solution to the diffusion equation to more solutions (ignoring the initial condition).

(a) The translate $u(x - y, t)$ of a solution $u(x, t)$ is another solution.

(b) Any derivative of a solution is a solution (derivative with respect to either x or t).

(c) A linear combination of solutions is a solution (by linearity).

(d) An integral of solutions is a solution. If $S(x, t)$ is a solution of the diffusion equation then so is $S(x - y, t)$ and so is

$$v(x, t) = \int_{-\infty}^{\infty} S(x - y) g(y)\, \mathrm{d}y$$

for any function $g(y)$ (provided the integral converges).

(e) If $u(x, t)$ is any solution to the diffusion equation then so is the **dilated** function

$$v(x, t) = u(\sqrt{a}x, at)$$

for any $a > 0$.

Note that (d) is really a limiting form of (c). To check (e) we simply apply the chain rule

$$\begin{aligned} v_t &= \frac{\partial(at)}{\partial t} u_t \\ &= a u_t. \end{aligned}$$

Similarly

$$\begin{aligned} v_x &= \frac{\partial(\sqrt{a}t)}{\partial x} u_x \\ &= \sqrt{a} u_x, \end{aligned}$$

so that

$$\begin{aligned} v_{xx} &= \sqrt{a}\sqrt{a} u_{xx} \\ &= a u_{xx}. \end{aligned}$$

We now write down a particular choice of $\phi(x)$.

Definition 11.1 (Heaviside step function)**.** *The* ***Heaviside step function*** *is defined as*

$$H(x) = \begin{cases} 1 & \textit{for } x > 0 \\ 0 & \textit{for } x < 0. \end{cases}$$

Note that the Heaviside step function does not change under dilation.

We are looking for a solution $Q(x,t)$ to the diffusion equation with the initial data of the Heaviside step function,

$$Q(x,0) = 1 \qquad \text{for } x > 0 \qquad Q(x,0) = 0 \qquad \text{for } x < 0.$$

There are three steps.

Step 1 We assume that $Q(x,t)$ has a very special form

$$Q(x,t) = g(p) \qquad \text{where} \qquad p = \frac{x}{\sqrt{4kt}}.$$

Why would we jump to this assumption? Well, there are two observations. A dilation of $Q(x,t)$ is still a solution to the wave equation and $H(x,t)$ is unchanged under dilations. It follows that any dilation of $Q(x,t)$ is a solution to the same PDE with the same auxiliary conditions. By uniqueness it follows that $Q(x,t)$ is unchanged under dilations.

Note that $x/\sqrt{t}$ is unchanged under dilations and in fact $Q(x,t)$ must be a function of $x/\sqrt{t}$. It is convenient to rescale and write Q as a function of p.

Step 2 We now figure out the ODE which g has to satisfy. If we apply the chain rule then we get

$$\begin{aligned} Q_t &= \frac{\mathrm{d}g}{\mathrm{d}p}\frac{\partial p}{\partial t} = -\frac{1}{2t}\frac{x}{\sqrt{4kt}}g'(p) \\ Q_x &= \frac{\mathrm{d}g}{\mathrm{d}p}\frac{\partial p}{\partial x} = \frac{1}{\sqrt{4kt}}g'(p) \\ Q_{xx} &= \frac{\mathrm{d}Q_x}{\mathrm{d}p}\frac{\partial p}{\partial x} = \frac{1}{4kt}g''(p) \\ 0 = Q_t - kQ_{xx} &= -\frac{1}{2t}\left(pg'(p) + \frac{1}{2}g''(p)\right). \end{aligned}$$

Putting all of this together we get

$$g'' + 2pg' = 0.$$

This gives us an integrating factor of

$$\exp(\int 2p\,\mathrm{d}p) = \exp(p^2).$$

This gives

$$g'(p) = c_1 \exp(-p^2)$$

so that

$$Q(x,t) = g(p) = c_1 \int_0^{\frac{x}{\sqrt{4kt}}} e^{-p^2}\,\mathrm{d}p + c_2.$$

Step 3 Now we turn this expression into an explicit formula. If t tends to zero from above then the upper limit increases in magnitude. If $x > 0$ then the upper limit approaches ∞ and Q approaches 1 and if $x < 0$ then the upper limit approaches $-\infty$ and Q approaches 0. Thus

$$1 = c_1 \int_0^{\infty} e^{-p^2}\,\mathrm{d}p + c_2 = c_1 \frac{\sqrt{\pi}}{2} + c_2$$

and

$$0 = c_1 \int_0^{-\infty} e^{-p^2}\,\mathrm{d}p + c_2 = -c_1 \frac{\sqrt{\pi}}{2} + c_2.$$

Thus

$$c_1 = \frac{1}{\sqrt{\pi}} \qquad \text{and} \qquad c_2 = \frac{1}{2}.$$

It follows that

$$Q(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{\frac{x}{\sqrt{4kt}}} e^{-p^2}\,\mathrm{d}p.$$

Step 4 Now we know Q we put $S = Q_x$. Then S is also a solution of the diffusion equation, by (b). Given ϕ, we define

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,\mathrm{d}y \qquad \text{for} \qquad t > 0.$$

Then u is a solution of the diffusion equation by (d).

We have

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} Q_x(x-y,t)\phi(y)\,\mathrm{d}y \\ &= -\int_{-\infty}^{\infty} Q_y(x-y,t)\phi(y)\,\mathrm{d}y \\ &= \Big[-Q(x-y)\phi(y)\Big]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} Q(x-y,t)\phi'(y)\,\mathrm{d}y. \end{aligned}$$

We assume that the first term is zero; in fact we assume that $\phi(y)$ is zero for y large. It follows that

$$\begin{aligned}
u(x,0) &= \int_{-\infty}^{\infty} Q(x-y,0)\phi'(y)\,\mathrm{d}y \\
&= \int_{-\infty}^{\infty} H(x-y)\phi'(y)\,\mathrm{d}y \\
&= \int_{-\infty}^{x} \phi'(y)\,\mathrm{d}y \\
&= \Big[\phi(y)\Big]_{-\infty}^{x} \\
&= \phi(x).
\end{aligned}$$

Thus $u(x,t)$ is indeed a solution to the diffusion equation and $u(x,0) = \phi(x)$.

Now

$$S = Q_x = \frac{1}{2\sqrt{\pi k t}} e^{-x^2/4kt} \qquad \text{for} \qquad t > 0.$$

Thus

$$u(x,t) = \frac{1}{2\sqrt{\pi k t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt}\phi(y)\,\mathrm{d}y.$$

is a solution of the diffusion equation, with initial condition $\phi(x)$. This is the analogue of d'Alembert's solution of the wave equation.

This gives the solution for $t > 0$. The formula does not make sense if $t = 0$.

The functions $S(x,t)$ has many different names: the **source function**, **Green's function**, **fundamental solution**, **Gaussian**, **propagator**, **diffusion kernel**.

$S(x,t)$ is defined for all x and for $t > 0$ and it is even, that is,

$$S(-x,t) = S(x,t).$$

If t is small then the graph looks like a spike, concentrated at the origin (a *delta function*), as t increases the graph resembles the familiar Gauss bell curve and if t is very large the graph looks almost flat.

Note that

$$\begin{aligned}
\int_{-\infty}^{\infty} S(x,t)\,\mathrm{d}x &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-q^2}\,\mathrm{d}q \\
&= 1,
\end{aligned}$$

where

$$q = \frac{x}{\sqrt{4kt}} \qquad \text{so that} \qquad \mathrm{d}q = \frac{\mathrm{d}x}{\sqrt{4kt}}.$$

If one looks at the graph of $S(x,t)$ for t small then notice if we cut out the tall thin spike then the rest of the function is close to zero:

$$\max_{|t|>\delta} S(x,t) \to 0 \qquad \text{as} \qquad t \to 0.$$

One can think of the solution above as a weighted average of contributions from corresponding to the initial value. In fact the integral is approximated by a Riemann sum

$$\sum_i S(x-y_i,t)\phi(y_i)\Delta y_i.$$

This is the average of the solutions $S(x-y_i,t)$ with weights $\phi(y_i)$. When t is small $S(x-y_i,t)$ is a spike concentrated over y_i and so the Riemann sum gives more weight to points y_i close to x and as t increases the weights even out.

The physical explanation of this is that $S(x-y,t)$ represents one unit (a gram say) of substance concentrated at time $t=0$ at the point y. As time progresses this substance diffuses, or spreads out. The term

$$S(x-y_i,t)\phi(y_i)\Delta y_i$$

reprents the diffusion of $\phi(y_i)\Delta y_i$ units of substance distributed over a small interval of width Δy_i centred around y_i. There is a similar picture for heat, $S(x-y,t)$ represents a hot spot centred at y at time $t=0$ which spreads out as time progresses.

It is rare that one can use the integral to give explicit formulae for concrete choices of $\phi(x)$. However it is not uncommon that one can express the answer in terms of the **error function**

$$\mathscr{E}\mathrm{rf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-p^2}\,\mathrm{d}p.$$

Note that

$$\mathscr{E}\mathrm{rf}(0) = 0 \qquad \text{and} \qquad \lim_{x\to\infty} \mathscr{E}\mathrm{rf}(x) = 1.$$

For example,

$$Q(x,t) = \frac{1}{2} + \frac{1}{2}\mathscr{E}\mathrm{rf}\left(\frac{x}{\sqrt{4kt}}\right).$$

Example 11.2. *Solve the diffusion equation with the initial condition*

$$u(x,0) = e^{-x}.$$

We use the formula

$$u(x,t) = \frac{1}{2\sqrt{\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} e^{-y}\,\mathrm{d}y.$$

Consider the exponent of e:

$$-\frac{x^2 - 2xy + y^2 + 4kty}{4kt}.$$

We complete the square in y:

$$-\frac{(y + 2kt - x)^2}{4kt} + kt - x.$$

Let

$$p = \frac{y + 2kt - x}{\sqrt{4kt}} \qquad \text{so that} \qquad \mathrm{d}p = \frac{\mathrm{d}y}{\sqrt{4kt}}.$$

It follows that

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{\pi}} e^{kt-x} \int_{-\infty}^{\infty} e^{-p^2}\,\mathrm{d}p \\ &= e^{kt-x}. \end{aligned}$$

Note that this solution grows rather than decays in time. This does not contradict the maximum principle, as the temperature of the left hand side of the rod is very hot, that is,

$$\lim_{x\to-\infty} u(x,0) = \infty.$$