1. First examples

A differential equation for a variable $u(x, y, \dots)$, which depends on the independent variables x, y, $\dots$, is an equation which involves u and some of its partial derivatives. Differential equations have traditionally played a huge role in the sciences and so they have been heavily studied in mathematics.

If u depends on x and y, there are two partial derivatives,

$$\frac{\partial u}{\partial x} \qquad \text{and} \qquad \frac{\partial u}{\partial y}.$$

Since partial differential equations might involve lots of variables and derivatives, it is convenient to introduce some more compact notation.

$$\frac{\partial u}{\partial x} = u_x \qquad \text{and} \qquad \frac{\partial u}{\partial y} = u_y.$$

Note that

$$u_{xx} = \frac{\partial^2 u}{\partial x^2} \qquad u_{xy} = \frac{\partial^2 u}{\partial x \partial y} \qquad \text{and} \qquad u_{yy} = \frac{\partial^2 u}{\partial y^2}.$$

Recall some basic facts about derivatives. First of all derivatives are local, meaning if you only change the function away from a point then the derivative is unchanged (contrast this with the integral, which is far from local; the area under the graph depends on the global behaviour). Secondly if all 2nd derivatives exist and they are continuous then the mixed partials are equal:

$$u_{xy} = u_{yx}.$$

Example 1.1. *Suppose that*

$$u(x, y) = \sin(xy).$$

Then

$$u_x = y\cos(xy) \qquad \text{and} \qquad u_y = x\cos(xy).$$

But then

$$u_{yx} = \cos(xy) - xy\sin(xy) \qquad \text{and} \qquad u_{xy} = \cos(xy) - xy\sin(xy).$$

Partial differential equations are in general very complicated and normally we can only make sense of the solutions of those equations which come from applications. If there are two space variables then we typically call them x and y but as usual t denotes a time variable and x a space variable.

Here are some examples of partial differential equations

(1) $u_x + u_y = 0$ (transport)
(2) $u_x + yu_y = 0$ (transport)

(3) $u_x + uu_y = 0$ (shock wave)
(4) $u_{xx} + u_{yy} = 0$ (Laplace's equation)
(5) $u_{tt} - u_{xx} + u^3 = 0$ (wave with interaction)
(6) $u_t + uu_x + u_{xxx} = 0$ (dispersive wave)
(7) $u_{tt} + u_{xxxx} = 0$ (vibrating bar)
(8) $u_t - iu_{xx} = 0$ (Schrödinger's wave equation)

The general form of a partial differential equation looks like

$$F(x, y, u, u_x, u_y, \dots) = 0.$$

Note that for ordinary differential equations, sometimes we blur the difference between dependent and independent variables. The easiest way to solve the ODE

$$\frac{\mathrm{d}u}{\mathrm{d}x} = u^3$$

is to write

$$\frac{\mathrm{d}x}{\mathrm{d}u} = u^{-3}$$

and integrate. We have no such luxury for PDEs.

The simplest invariant one can attach to a partial differential equation is its order, meaning the largest order of a partial derivative.

The first, second and third equations have order one; the fourth, fifth and eighth have order two; the sixth order three; the seventh order four.

The most tractable PDEs are linear. Suppose that we write the equation in the form

$$\mathscr{L}\, u = 0.$$

Here $\mathscr{L}$ is an operator, typically involving derivatives. For the first equation, we would put

$$\mathscr{L} = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$$

so that when we apply $\mathscr{L}$ to u we get the LHS of equation one. We say that $\mathscr{L}$ is **linear** if

$$\mathscr{L}(u + v) = \mathscr{L}(u) + \mathscr{L}(v) \qquad \text{and} \qquad \mathscr{L}(cu) = c\,\mathscr{L}(u)$$

for all choices of functions u and v and scalars c. The adage to remember is "double the input, double the output"and perhaps not surprisingly linear operators are quite common in practice.

We say a PDE is **linear** if it is of the form

$$\mathscr{L}(u) = 0,$$

for a linear operator $\mathscr{L}$.

(1), (2), (4), (7) and (8) are linear equations. (3), (5) and (6) are not linear. (3) is not linear because of the uu_y term, (5) is not linear because of the u^3 term, and (6) is not linear because of the uu_x term.

We sometimes refer to a linear equation

$$\mathscr{L}(u) = 0,$$

as a **homogeneous linear equation**. If an equation has the form

$$\mathscr{L}(u) = g,$$

where g is a function of the independent variables and $\mathscr{L}$ is a linear operator then we say the equation is an **inhomogeneous linear equation**. For example,

$$\log(xy)u_x + \tan(x^2 + y^2)u_y = e^{y^3 - x}$$

is an inhomogeneous linear equation.

We will focus almost exclusively on linear equations, in fact linear equations with constant coefficients. The key property of a linear equation is that if u and v are solutions of the linear equation then so is $u + v$:

$$\begin{aligned}\mathscr{L}(u+v) &= \mathscr{L}(u) + \mathscr{L}(v)\\ &= 0 + 0\\ &= 0.\end{aligned}$$

More generally, if $u_1, u_2, \ldots, u_n$ are solutions of a linear equation and $c_1, c_2, \ldots, c_n$ are scalars then

$$c_1u_1(x) + c_2u_2(x) + \cdots + c_nu_n(x)$$

is also a solution. This is sometimes called the **principle of superposition**. Another basic property of linear operators is that the sum of a solution to an inhomogeneous linear equation plus a solution to the underlying homogeneous linear equation is a solution to the original inhomogeneous linear equation.

Now let's look at some examples.

Example 1.2. *Solve*

$$u_{xx} = 0$$

where $u(x, y)$ is a function of x and y.

We solve this the usual way, by integrating both sides with respect to x. This removes one partial derivative from the LHS and introduces a constant of integration on the RHS.

However this hides a subtlety, the constant of integration is in fact an arbitrary function of y,

$$u_x = f(y).$$

There are a number of ways to see that we get a function of y on the RHS. For each value of y we get an ODE involving x. If we integrate this ODE we get a constant of integration, but as we vary y, there is no reason to expect this constant of integration to remain fixed. Perhaps even more compelling, if we start with the function $f(y)$ and we differentiate it once with respect to x then we always get zero, as y is independent of x.

If we integrate with respect to x one more time then we get

$$u(x,y) = f(y)x + g(y),$$

is the general solution to the PDE

$$u_{xx} = 0$$

Here $f(y)$ and $g(y)$ are arbitrary functions of y.

Example 1.3. *Solve*

$$u_{xx} - u = 0$$

where $u(x,y)$ is a function of x and y.

Again, we just solve the corresponding ODE, taking account of the fact that the constants of integration are in fact arbitrary functions of y,

$$u(x,y) = f(y)e^x + g(y)e^{-x}.$$

Here $f(y)$ and $g(y)$ are again arbitrary functions of y.

Example 1.4. *Solve*

$$u_{xy} = 0.$$

If we integrate with respect to x we get

$$u_y = f(y).$$

Now if integrate with respect to y we get

$$u = F(y) + G(x),$$

where

$$F' = f.$$

Note that 2nd order PDEs all have two arbitrary functions in their general solution. In the examples we gave, these arbitrary functions are functions of one variable. So the function $u(x,y)$ is ambiguous but at least the ambiguity is a function of one variable.

In fact functions of two variables are far more complicated than functions of one variable. This is obvious from the graph of a function of two variables, but it is also obvious if you think about how to approximate functions. If you wanted to approximate a function of one variable, you might pick a hundred equally spaced points $x_1, x_2, \ldots, x_{100}$ and write down a table with two columns, x_i and the value $f(x_i)$.

If you carried out the analagous procedure for a function $f(x, y)$ of two variables, you would take a grid, with $100 \times 100 = 10^4$ points. The table would then contain three columns, a value for x_i a value for y_i and the value of the function $f(x_i, y_i)$. The important thing is that we need 10^4 rows to get the same level of approximation.