FINAL EXAM
MATH 110A, UCSD, AUTUMN 18

You have three hours.

There are 9 problems, and the total number of points is 105. Show all your work. *Please make your work as clear and easy to follow as possible.*

Name:______________________________

Signature:______________________________

Student ID #:______________________________

Problem	Points	Score
1	15	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	20	
9	10	
10	10	
11	10	
12	10	
13	10	
Total	105	

1. (15pts) (i) *Give the definition of an odd function.*

f is odd if
$$f(-x) = -f(x).$$

(ii) *Write down the wave equation.*

$$u_{tt} = c^2 u_{xx}.$$

(iii) *Write down the (general) Fourier sine series for the interval* $(0, l)$.

$$\sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}.$$

2. (10pts) *Which of the following operators are linear?*
(a)

$$\mathscr{L}\, u = \sqrt{1+x^2}(\cos y)u_x + u_{yxy} - \tan^{-1}(x/y)u.$$

This is linear. The general form of a linear operator is a combination of arbitrary order partial derivatives of u with respect to x and y with coefficients which are functions of x and y.

(b)

$$\mathscr{L}\, u = u_x + u_y + 1.$$

This is not linear.

$$\begin{aligned}\mathscr{L}\, 2u &= 2u_x + 2u_y + 1 \\ &= 2u_x + 2u_y + 2 - 1 \\ &= 2\,\mathscr{L}\, u - 1 \\ &\neq 2\,\mathscr{L}\, u.\end{aligned}$$

3. (10pts) (a) *Find the general solution of*

$$u_x + yu_y = 0,$$

The solution is constant along the characteristic curve with equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{1}.$$

Separating the variables gives

$$y = ce^x$$

where c is a constant. Thus the general solution is an arbitrary function of $c = ye^{-x}$,

$$u(x, y) = f(ye^{-x}).$$

(b) *Show that one cannot solve*

$$u_x + yu_y = 0$$

subject to the boundary condition $u(x, 0) = x$.

We want to choose f so that

$$\begin{aligned} x &= u(x, 0) \\ &= f(e^{-x}). \end{aligned}$$

Let $w = e^{-x}$. Then

$$x = -\log w$$

and

$$f(w) = -\log w.$$

This gives the solution

$$u(x, y) = -\log(ye^{-x}).$$

However the logarithm function is not defined at zero. In fact

$$\lim_{t \to 0^+} \log t = -\infty$$

and so this function does not have the correct behaviour along the boundary.

4. (10pts) *Find the regions in the xy-plane where the equation*

$$(1+x)u_{xx} + 2xyu_{xy} - y^2u_{yy} = 0$$

is elliptic, hyperbolic and parabolic.

The coefficients are $1+x$, xy and $-y^2$. We consider the sign of the discriminant

$$\begin{aligned}\mathscr{D} &= (xy)^2 - (1+x)(-y^2) \\ &= x^2y^2 + y^2 + xy^2 \\ &= y^2(x^2+x+1).\end{aligned}$$

y^2 is always at least zero. If $y=0$ the discriminant is zero. On the other hand, x^2+x+1 is always positive.
If $y=0$ then the PDE is elliptic. If $y \neq 0$ the PDE is hyperbolic.

5. (10pts) *Solve*

$$u_t = ku_{xx} \qquad u(x,0) = e^{-x} \qquad \text{and} \qquad u(0,t) = 0$$

on the half line $0 < x < \infty$.

We want to solve the diffusion equation on the half line. Let ϕ_{odd} be the odd extension of $\phi(x) = e^{-x}$ to the whole line. Then

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} S(x-y,t)\phi_{\text{odd}}(y)\,\mathrm{d}y \\ &= \int_{0}^{\infty} (S(x-y,t) + S(x+y,t))\phi(y)\,\mathrm{d}y \\ &= \frac{1}{\sqrt{4\pi kt}} \int_{0}^{\infty} (e^{(x-y)^2/4\pi kt} + e^{(x+y)^2/4\pi kt})e^{-y}\,\mathrm{d}y \end{aligned}$$

We complete the square in both integrals. Note that

$$-(x+y)^2 = -(-(-x)-y)^2.$$

The exponents of the exponentials in the two integrals are

$$-\frac{(y+2kt-x)^2}{4kt} + kt - x \qquad \text{and} \qquad -\frac{(y+2kt+x)^2}{4kt} + kt + x.$$

To get the second expression just flip the sign of x. We make the change of variables

$$p = \frac{y+2kt-x}{\sqrt{4kt}} \qquad \text{and} \qquad q = \frac{y+2kt+x}{\sqrt{4kt}},$$

so that

$$\mathrm{d}p = \frac{\mathrm{d}y}{\sqrt{4kt}} \qquad \text{and} \qquad \mathrm{d}q = \frac{\mathrm{d}y}{\sqrt{4kt}}.$$

Thus

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{\pi}}e^{kt-x}\int_{(2kt-x)/\sqrt{4kt}}^{\infty} e^{-p^2}\,\mathrm{d}p - \frac{1}{\sqrt{\pi}}e^{kt+x}\int_{(2kt+x)/\sqrt{4kt}}^{\infty} e^{-q^2}\,\mathrm{d}q \\ &= \frac{1}{2}(e^{kt-x} - e^{kt+x}) - e^{kt-x}\,\mathscr{E}\mathrm{rf}((2kt-x)/\sqrt{4kt}) + e^{kt+x}\,\mathscr{E}\mathrm{rf}((2kt+x)/\sqrt{4kt}). \end{aligned}$$

6. (10pts) *Solve*

$$u_{tt} = 9u_{xx}, \quad 0 < x < \infty, \quad u(0,t) = 0, \quad u(x,0) = 1, \quad u_t(x,0) = 0$$

using the reflection method. This solution has a singularity; find its location.

Let $\phi(x) = 1$ and $\psi(x) = 0$ on the half line $0 < x < \infty$. Then the odd extensions are $\psi_{\text{odd}} = 0$ and

$$\phi_{\text{odd}}(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0. \end{cases}$$

We apply d'Alembert's formula

$$u(x,t) = \frac{1}{2}\left(\phi_{\text{odd}}(x-3t) + \phi_{\text{odd}}(x+3t)\right).$$

There are two cases. If $x < 3t$ then we get

$$u(x,t) = 1 + 1 = 2.$$

If $x > 3t$ then we get

$$u(x,t) = 1 - 1 = 0.$$

Thus

$$u(x,t) = \begin{cases} 2 & \text{if } x < 3t \\ 0 & \text{if } x > 3t. \end{cases}$$

There is a singularity along the line $x = 3t$.

7. (10pts) *Solve*

$$u_{tt} = c^2 u_{xx} + e^{ax} \qquad u(x,0) = 0 \qquad \text{and} \qquad u_t(x,0) = 0.$$

We want to solve the wave equation with a source on the whole interval. We apply the formula

$$\begin{aligned}
u(x,t) &= \frac{1}{2c} \iint_\Delta f \\
&= \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} e^{ay} \, \mathrm{d}y \, \mathrm{d}s \\
&= \frac{1}{2ac} \int_0^t \Big[e^{ay} \Big]_{x-c(t-s)}^{x+c(t-s)} \mathrm{d}s \\
&= \frac{1}{2ac} \int_0^t e^{a(x+c(t-s))} - e^{a(x-c(t-s))} \, \mathrm{d}s \\
&= \frac{1}{2a^2c^2} \Big[-e^{a(x+c(t-s))} - e^{a(x-c(t-s))} \Big]_0^t \\
&= \frac{1}{2a^2c^2} \left(-e^{ax} - e^{ax} + e^{a(x+ct)} + e^{a(x-ct)} \right) \\
&= \frac{1}{2a^2c^2} \left(e^{a(x+ct)} + e^{a(x-ct)} - 2e^{ax} \right).
\end{aligned}$$

One can indeed check that this satisfies all of the conditions.

8. (20pts) *Find*
(a) The Fourier sine series on the interval $(0, l)$ for $\phi(x) = 1$.

We compute

$$\begin{aligned} A_m &= \frac{2}{l}\int_0^l \sin\frac{m\pi x}{l}\,\mathrm{d}x \\ &= \frac{2}{m\pi}\left[-\cos\frac{m\pi x}{l}\right]_0^l \\ &= \frac{2}{m\pi}\left(-\cos m\pi + 1\right) \\ &= \begin{cases} \frac{4}{m\pi} & m \text{ is odd} \\ 0 & m \text{ is even.} \end{cases} \end{aligned}$$

Thus

$$1 = \frac{4}{\pi}\left(\sin\frac{\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + \frac{1}{5}\sin\frac{5\pi x}{l} + \dots\right)$$

(b) *The Fourier sine series on the interval* $(0, l)$ *for* $\phi(x) = x$.

We compute

$$\begin{aligned} A_m &= \frac{2}{l}\int_0^l x\sin\frac{m\pi x}{l}\,\mathrm{d}x \\ &= \frac{2}{m\pi}\left[-x\cos\frac{m\pi x}{l} + \frac{l}{m\pi}\sin\frac{m\pi x}{l}\right]_0^l \\ &= \frac{2l}{m\pi}(-1)^{m+1}. \end{aligned}$$

Thus

$$x = \frac{2l}{\pi}\left(\sin\frac{\pi x}{l} - \frac{1}{2}\sin\frac{2\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + \dots\right).$$

(c) *The Fourier cosine series on the interval* $(0, l)$ *for* $\phi(x) = x^2/2$.

First we integrate the Fourier sine series for x to get

$$\frac{x^2}{2} = c + \frac{2l^2}{\pi^2}\left(-\cos\frac{\pi x}{l} + \frac{1}{4}\cos\frac{2\pi x}{l} - \frac{1}{9}\cos\frac{3\pi x}{l} + \dots\right).$$

Now we integrate both sides from 0 to l to determine c.

$$cl = \frac{l^3}{6}.$$

Thus

$$\frac{x^2}{2} = \frac{l^2}{6} + \frac{2l^2}{\pi^2}\left(-\cos\frac{\pi x}{l} + \frac{1}{4}\cos\frac{2\pi x}{l} - \frac{1}{9}\cos\frac{3\pi x}{l} + \dots\right).$$

(d) The Fourier series on the interval $(-l, l)$ for $\phi(x) = x^3/6$.

First we integrate the Fourier cosine series for $x^2/2$ to get the Fourier sine series on $(0, l)$. The constant of integration is zero as both sides are odd.

$$\begin{aligned}\frac{x^3}{6} &= \frac{l^2 x}{6} + \frac{2l^3}{\pi^3}\left(-\sin\frac{\pi x}{l} + \frac{1}{8}\sin\frac{2\pi x}{l} - \frac{1}{27}\sin\frac{3\pi x}{l} + \dots\right) \\ &= \frac{l^3}{3\pi}\left(\sin\frac{\pi x}{l} - \frac{1}{2}\sin\frac{2\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + \dots\right) + \frac{2l^3}{\pi^3}\left(-\sin\frac{\pi x}{l} + \frac{1}{8}\sin\frac{2\pi x}{l} - \frac{1}{27}\sin\frac{3\pi x}{l} + \right. \\ &= \frac{l^3}{3\pi^3}\left((\pi^2 - 6)\sin\frac{\pi x}{l} + (\frac{3}{4} - \frac{\pi^2}{2})\sin\frac{2\pi x}{l} + (\frac{\pi^2}{3} - \frac{2}{9})\sin\frac{3\pi x}{l} + \dots\right).\end{aligned}$$

Now observe that since both sides are odd, this is also the Fourier series for $x^3/6$ on the whole interval $(-l, l)$.

9. (10pts) *Consider a metal rod* $0 < x < l$*, insulated along its sides but not at its ends, which is initially at temperature* $= 1$*. Suddenly both ends are plunged into a bath a temperature* $= 0$*. Write the formula for the differential equation, boundary conditions and initial condition. Give the formula for the temperature* $u(x,t)$ *at later times.*

We want to solve the diffusion equation

$$u_t = ku_{xx}$$

for the interval $0 < x < l$ with initial condition $u(x,0) = 1$ and boundary conditions $u(0,t) = u(l,t) = 0$.
We want to find the eigenfunctions $X'' = -\lambda X$ such that $X(0) = X(l) = 0$. These are

$$X_n = \sin \frac{n\pi x}{l}.$$

The separated solutions are then of the form

$$u_n(x,t) = e^{-kn^2\pi^2 kt/l^2} \sin \frac{n\pi x}{l}.$$

Turning these into a series we get

$$u(x,t) = \sum_{n \geq 1} A_n e^{-kn^2\pi^t/l^2} \sin \frac{n\pi x}{l}.$$

This satisfies the boundary conditions but not the initial conditions. If we plug in $t = 0$ we get

$$1 = \sum_{n \geq 1} A_n \sin \frac{n\pi x}{l}.$$

The coefficients $A_1, A_2, \dots$ are given by 8(i). It follows that

$$u(x,t) = \frac{4}{\pi}\left(e^{-k\pi^2 t/l^2} \sin \frac{\pi x}{l} + \frac{1}{3} e^{-9k\pi^2 t/l^2} \sin \frac{3\pi x}{l} + \frac{1}{5} e^{-25k\pi^2 t/l^2} \sin \frac{5\pi x}{l} + \dots\right).$$

Bonus Challenge Problems

10. (10pts) *Find the general solution of*

$$3u_{tt} + 10u_{xt} + 3u_{xx} = \sin(x + t).$$

We have

$$\begin{aligned} 3u_{tt} + 10u_{xt} + 3u_{xx} &= 3\partial_x^2 u + 10\partial_x\partial_t u + 3\partial_t^2 u \\ &= (3\partial_x^2 + 10\partial_x\partial_t + 3\partial_t^2)u \\ &= (3\partial_x + \partial_t)(\partial_x + 3\partial_t)u. \end{aligned}$$

Therefore we have to solve

$$3v_x + v_t = \sin(x + t) \qquad \text{where} \qquad v = u_x + 3u_t.$$

The homogeneous version of the first equation has general solution

$$v(x, t) = h(3t - x),$$

where h is an arbitrary differentiable function of one variable. Therefore

$$v(x, t) = h(3t - x) - \cos(x + t)/4$$

is the general solution of the original inhomogeneous equation. Thus we now just need to solve

$$u_x + 3u_t = h(3t - x) - \cos(x + t)/4.$$

A particular solution is given by

$$u(x, t) = f(3t - x) - \sin(x + t)/16,$$

where

$$f' = h/8.$$

The associated homogeneous equation is

$$u_x + 3u_t = 0.$$

This has general solution

$$u(x, t) = g(3x - t).$$

Thus the general solution to the original inhomogeneous equation is

$$u(x, t) = f(3t - x) + g(3x - t) - \sin(x + t)/16,$$

where f and g are arbitrary twice differentiable functions.

11. (10pts) *Show that the (weak) maximum principle holds for the diffusion equation on the rectangle* $0 \leq x \leq l$, $0 \leq t \leq T$.

See lecture 10.

12. (10pts) *Dervive the formula for inhomogeneous wave equation on the whole line*

$$u_{tt} - c^2 u_{xx} = f(x,t) \qquad u(x,0) = \phi(x) \qquad \text{and} \qquad u_t(x,0) = \psi(x).$$

See lecture 16 or model answers 7, 3.4.6 or 3.4.12. In addition, the book also has many other ways to derive this formula.

13. (10pts) *Calculate*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}.$$

We use the answer to 8 (c).

$$\frac{x^2}{2} = \frac{l^2}{6} + \frac{2l^2}{\pi^2}\left(-\cos\frac{\pi x}{l} + \frac{1}{4}\cos\frac{2\pi x}{l} - \frac{1}{9}\cos\frac{3\pi x}{l} + \dots\right).$$

If we set $x = 0$ then we get

$$0 = \frac{l^2}{6} + \frac{2l^2}{\pi^2}\left(-1 + \frac{1}{4} - \frac{1}{9} + \dots\right).$$

Solving for -1 times the expression in parentheses gives

$$1 - \frac{1}{4} + \frac{1}{9} + \frac{1}{25} + \cdots = \frac{\pi^2}{12}.$$

Thus

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$