PRACTICE PROBLEMS FOR THE SECOND MIDTERM

1. Give the definition of:
   - (i) $\ll$ and $\gg$.
   - (ii) $\liminf$ and $\limsup$.
   - (iii) $\mathrm{li}(x)$.
   - (iv) The integer nearest to x.
   - (v) $\vartheta(x)$.
   - (vi) Riemann zeta-function $\zeta(s)$.
   - (vii) completely multiplicative.
   - (viii) Dirichlet series.

2. Show that

(a)

$$\pi(x) \leq r + x\prod_{i=1}^{r}\left(1-\frac{1}{p_i}\right) + 2^r,$$

where $p_1, p_2, \ldots, p_r$ are the first r primes.

(b) If $x \geq 2$ then

$$\prod_{i=1}^{r}\left(1-\frac{1}{p_i}\right) < \frac{1}{\log x}.$$

(c)

$$\pi(x) \ll \frac{x}{\log\log x}.$$

3. Show that

$$\sum_{p\leq x} p^{-1} > \frac{1}{2}\log\log x.$$

4. Assuming that

$$\sum_{p\leq x}\frac{\log p}{p} = \log x + O\left(\frac{\log x}{\log\log x}\right)$$

show that

$$\sum_{p\leq x}\frac{1}{p} \sim \log\log x.$$

5. Derive the prime number theorem from the relation

$$\vartheta(x) \sim x.$$

6. Show that

$$\sum(-1)^k k^{-s}$$

converges for $s > 0$.

7. Show that

$$\mathrm{li}(x) = \frac{x}{\log x} + \frac{1!x}{\log^2 x} + \frac{2!x}{\log^3 x} + \cdots + \frac{(n-1)!x}{\log^n x} + O\left(\frac{x}{\log^{n+1} x}\right).$$

8. Assuming that there are constants c_1 and c_2 such that

$$c_1 \frac{x}{\log x} < \pi(x) < c_2 \frac{x}{\log x},$$

show that there are constants c_3 and c_4 such that

$$c_3 r \log r < p_r < c_4 r \log r.$$