PRACTICE PROBLEMS FOR THE FIRST MIDTERM

1. Give the definition of:

(i) τ.
(ii) σ.
(iii) $\pi(x)$.
(iv) a multiplicative function.
(v) a perfect number.
(vi) a Mersenne prime.
(vii) The Möbius function.
(viii) the round down.
(ix) the fractional part.
(x) $f(x) = O(g(x))$.
(xi) $f(x) = o(g(x))$.
(xii) $f(x) \sim g(x)$.
(xiii) Euler's constant.

2. Show that if $f\colon \mathbb{N} \longrightarrow \mathbb{C}$ is a multiplicative function then

$$F(n) = \sum_{d|n} f(d)$$

is a multiplicative function.

3. Show that if n is an even perfect number then $n = 2^{p-1}(2^p - 1)$, where $2^p - 1$ is a Mersenne prime.

4. Show that the number of ordered pairs pairs of natural numbers whose lowest common multiple is n is $\tau(n^2)$.

5. Show that if $d|n$ and $(n, r) = 1$ then the number of solutions, modulo n, of

$$x \equiv r \mod d \qquad \text{where} \qquad (x, n) = 1,$$

is

$$\frac{\varphi(n)}{\varphi(d)} = \frac{n}{d} \prod_{p|n, p\nmid d} \left(1 - \frac{1}{p}\right).$$

6. Show that if f and F are as in (2) then (a)

$$f(n) = \sum_{d|n} \mu(d) F(n/d).$$

(b) if F is multiplicative then f is multiplicative.

7. Let $f(x, n)$ be the number of integers less than or equal to x and coprime to n. Prove that

(a)

$$\sum_{d|n} f(\frac{x}{d}, \frac{n}{d}) = \llcorner x \lrcorner$$

(b)

$$f(x,n) = \sum_{d|n} \mu(d) \llcorner \frac{x}{d} \lrcorner.$$

8. Show that

$$\prod_{p \le x} \left(1 - \frac{1}{p}\right) < \frac{1}{\log x}.$$