MODEL ANSWERS TO THE NINTH HOMEWORK

6.11.6. We apply partial summation to

$$\lambda_n = n \qquad c_n = f(n) \qquad \text{and} \qquad \frac{1}{g(x)}.$$

We get

$$\sum_{n\leq x} \frac{f(x)}{g(x)} = \frac{\sum_{n\leq x} f(n)}{g(x)} + \int_1^x \frac{g'(t)\sum_{n\leq t} f(n)}{g(t)^2}\,\mathrm{d}t.$$

As

$$\sum_{n\leq x} f(n) \sim xg(x)$$

we have

$$\begin{aligned}\int_1^x \frac{g'(t)\sum_{n\leq t} f(n)}{g(t)^2}\,\mathrm{d}t &= \int_1^x \frac{tg'(t)+o(tg'(t))}{g(t)}\,\mathrm{d}t \\ &= \int_1^x \frac{o(g(t))}{g(t)}\,\mathrm{d}t \\ &= \int_1^x o(1)\,\mathrm{d}t \\ &= o(x).\end{aligned}$$

Hence

$$\sum_{n\leq x} \frac{f(x)}{g(x)} = x + o(x),$$

so that

$$\sum_{n\leq x} \frac{f(x)}{g(x)} \sim x.$$

6.11.7. Since $\tau(mn) = \tau(m)\tau(n)$ if m and n are coprime, we may assume that $m = p^e$ and $n = p^f$ are powers of the same prime p. In this case

$$\begin{aligned}\tau(m)\tau(n) &= \tau(p^e)\tau(p^f)\\ &= (1+e)(1+f)\\ &= 1+e+f+ef\\ &\geq 1+ef\\ &= \tau(p^{e+f})\\ &= \tau(mn).\end{aligned}$$

We have

$$\begin{aligned}\sum_{n\leq x}\tau^2(n) &= \sum_{n\leq x}\tau(n)\sum_{d|n}1\\ &= \sum_{d_1d_2\leq x}\tau(d_1d_2)\\ &\leq \sum_{d_2\leq x}\sum_{d_1\leq x/d_2}\tau(d_1)\tau(d_2)\\ &= \sum_{d_2\leq x}\tau(d_2)\sum_{d_1\leq x/d_2}\tau(d_1)\\ &= \sum_{d\leq x}\tau(d)(x/d\log(x/d)+O(x/d))\\ &= x\log x\sum_{d\leq x}\frac{\tau(d)}{d}(1+O\left(\frac{1}{\log x}\right))\\ &= 1/2x\log x(\log^2 x+O(\log x)(1+O\left(\frac{1}{\log x}\right))\\ &= 1/2x\log^3 x\left(1+O\left(\frac{1}{\log x}\right)\right)^2\end{aligned}$$

Thus

$$\sum_{n\leq x}\tau^2(n)\ll x\log^3 x.$$

6.11.10. The number of lattice points is

$$\begin{aligned}\sum_{|x|\leq\sqrt{n}}\sum_{|y|\leq\sqrt{n-x^2}}1 &= 2\sum_{|x|\leq\sqrt{n}}\lfloor\sqrt{n-x^2}\rfloor\\ &= 4\sum_{0\leq x\leq\sqrt{n}}\sqrt{n-x^2}+O(\sqrt{n}).\end{aligned}$$

To estimate the last sum we use Riemann sums:

$$\sum_{0<x\le\sqrt{n}} \sqrt{n-x^2} \le \int_0^n \sqrt{n-t^2}\,\mathrm{d}t \le \sum_{0\le x<\sqrt{n}} \sqrt{n-x^2}.$$

As

$$\int_0^n \sqrt{n-t^2}\,\mathrm{d}t = \pi n$$

and the difference between the upper and low sum is bounded by $\sqrt{n}$, the number of lattice points inside the circle $x^2+y^2\le n$ is

$$\pi n + O(\sqrt{n}).$$

Here is a more geometric argument. For each lattice point p in the circle $x^2+y^2\le n$ imagine placing a unit square with vertices at the lattice points such that p is the southwest corner. The area covered is the number of lattice points in the circle. The circle $x^2+y^2=(\sqrt{n}+2)^2$ completely covers the squares we put down and the circle $x^2+y^2=(\sqrt{n}-2)^2$ is completely contained in the squares we put down. The difference in the areas of these two circles is

$$\begin{aligned}\pi(\sqrt{n}+2)^2-\pi(\sqrt{n}-2)^2 &= \pi(n+4\sqrt{n}+4-n-4\sqrt{n}+4)\\ &= O(\sqrt{n}),\end{aligned}$$

and of course the area of the circle $x^2+y^2=n$ is πn.

6.11.13. (a) We have

$$\begin{aligned}\sum_{a=1}^{m} a^r \sum_{d|a,d|m} \mu(d) &= \sum_{a=1}^{m} a^r \sum_{d|(a,m)} \mu(d)\\ &= \sum_{a=1}^{m} a^r M((a,m))\\ &= \sum_{a=1,(a,m)=1}^{m} a^r\\ &= F_r(m).\end{aligned}$$

(b) We first approximate the sum of the rth powers of the first n natural numbers. We use the Euler-MacLaurin formula with $f(x)=x^r$ and $m=0$. We have

$$\begin{aligned}\sum_{k=0}^{n} k^r &= \int_0^n x^r\,\mathrm{d}x + \frac{1}{2}n^r + \int_0^n r x^{r-1}\eta(x)\,\mathrm{d}x\\ &= \frac{n^{r+1}}{r+1} + O(n^r).\end{aligned}$$

We have

$$
\begin{aligned}
F_r(m) &= \sum_{a=1}^{m} a^r \sum_{d|a, d|m} \mu(d) \\
&= \sum_{\substack{d_1 d_2 \leq m \\ d_2 | m}} \mu(d_2)(d_1 d_2)^r \\
&= \sum_{d_2|m} \sum_{d_1 \leq m/d_2} \mu(d_2) d_1^r d_2^r \\
&= \sum_{d_2|m} \mu(d_2) d_2^r \sum_{d_1 \leq m/d_2} d_1^r \\
&= \sum_{d|m} \mu(d) d_1^r \left(\frac{(m/d)^{r+1}}{r+1} + O((m/d)^r) \right) \\
&= m^r \sum_{d|m} \mu(d) \left(\frac{m}{(r+1)d} + O(\frac{1}{d}) \right) \\
&= \frac{m^r \varphi(m)}{r+1} + O(m^r \tau(m)).
\end{aligned}
$$

6.12.1. Recall that

$$
\frac{t}{1+t} < \log(1+t) < t \qquad \text{for} \qquad t > -1.
$$

and $t \neq 0$. It follows that

$$
\begin{aligned}
a \log \left(1 - \frac{1}{p} \right) &> \frac{-\frac{a}{p}}{1 - \frac{1}{p}} \\
&= \frac{p}{p-1} \left(-\frac{a}{p} \right) \\
&> -\frac{a}{p} \\
&> \log \left(1 - \frac{a}{p} \right),
\end{aligned}
$$

for $p > a$. Taking the exponential of both sides gives

$$
\left(1 - \frac{a}{p} \right) < \left(1 - \frac{1}{p} \right)^a.
$$

Thus

$$\prod_{a<p<x}\left(1-\frac{a}{p}\right) < \prod_{a<p<x}\left(1-\frac{1}{p}\right)^a$$
$$\ll \prod_{p<x}\left(1-\frac{1}{p}\right)^a$$
$$\ll \log^{-a} x.$$

6.12.2. Our goal is to prove that

$$\pi(x) \ll Q(x) \qquad \text{where} \qquad Q(x) = \frac{x\log\log x}{\log x}.$$

Pick an integer $y \leq \sqrt{x}$ and let

$$R = \prod_{p\leq y} p.$$

Let $A(x,y)$ be the number of natural numbers up to x coprime to R. Let $r = \pi(y)$. Then

$$\pi(x) \leq r + A(x,y).$$

By inclusion-exclusion, we have

$$A(x,y) = S_0 - S_1 + S_2 + \dots$$

where S_i is the number of integers divisible by at least i factors of R. We claim that

$$A(x,y) \leq S_0 - S_1 + \dots + S_{2k},$$

for any even index $0 < 2k \leq \pi(y)$. Pick $n \leq x$ and suppose that n is divisible by m factors of R. Then the contribution of n to the RHS is

$$C(n) = 1 - \binom{m}{1} + \binom{m}{2} - \dots + \binom{m}{2k}.$$

If $m = 0$ then n is counted once on the LHS and once on the RHS, which is correct. If $m > 0$ then we just need to check that $C(n) \geq 0$, which was checked in the proof of Brun's theorem.
We have

$$S_l = \sum_{d=p_1p_2\dots p_l|R} \left\lfloor\frac{x}{d}\right\rfloor.$$

As before we estimate this by removing the round downs. If we remove the round downs we introduce an error of order at most

$$\sum_{l\leq 2k}\binom{r}{l} \leq 2^r.$$

If we multiply out

$$\prod_{p\le y}\left(1-\frac{1}{p}\right)$$

then we get

$$R_0 - R_1 + R_2 + \dots$$

where

$$R_l = \sum_{d=p_1p_2\dots p_l|R}\frac{1}{d}.$$

Putting all of this together, we get

$$A(x,y) \le r + 2^k + x\prod_{p\le y}\left(1-\frac{1}{p}\right) + xV_k,$$

where

$$V_k = -R_{2k+1} + R_{2k+2} + \dots.$$

Now

$$R_l \le \frac{1}{l!}\left(\sum_{p\le y}\frac{1}{p}\right)^l.$$

Therefore

$$|v_k| < \sum_{2k<l\le r}\frac{1}{l!}\left(\sum_{p\le y}\frac{1}{p}\right)^l$$

$$\sum_{l>2k}\frac{1}{l!}(\log\log y + c)^l,$$

for some constant $c > 0$. As

$$l! > \left(\frac{l}{e}\right)^l,$$

we have

$$|V_k| < \sum_{l>2k}\left(\frac{e\log\log y + ec}{l}\right)^l.$$

If we choose

$$k = \llcorner 3\log\log y \lrcorner$$

then

$$k > e\log\log y + ec,$$

for y large and in this case

$$\begin{aligned}|V_k| &< \sum_{l>2k} 2^{-l} \\ &= 2^{-2k} \\ &< 2^{-6\log\log y} \\ &< \frac{1}{\log^4 y}.\end{aligned}$$

On the other hand,

$$\prod_{p\le y}\left(1-\frac{1}{p}\right) \ll \frac{1}{\log y}.$$

Therefore

$$\begin{aligned}\pi(x) &\ll \pi(y) + 2^{3\log\log y} + \frac{x}{\log y} + \frac{x}{\log^4 y} \\ &\ll \frac{y}{\log\log y} + \log^2 y + \frac{x}{\log y} \\ &\ll \frac{y}{\log\log y} + \frac{x}{\log y}\end{aligned}$$

If we take

$$y = x^{1/\log\log x}.$$

then

$$\pi(x) \ll \frac{x\log\log x}{\log x}.$$