## MODEL ANSWERS TO THE EIGHTH HOMEWORK

6.10.1. Let

$$n_k = \prod_{i=1}^{k} p_i$$

be the product of the first $k$ primes. Then

$$\log n_k = \sum_{i=1}^{k} \log p_i < c p_k$$

for some constant $c$. Hence

$$\log \log n_k < \log c + \log p_k,$$

and so

$$\frac{n_k}{\log \log n_k} = O\left(\frac{n_k}{\log p_k}\right).$$

On the other hand,

$$\begin{aligned} \varphi(n_k) &= n_k \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right) \\ &= O\left(\frac{n_k}{\log n_k}\right). \end{aligned}$$

As $\log p_k \ll \log n_k$ it follows that

$$\varphi(n_k) \ll \frac{n_k}{\log \log n_k}.$$

6.10.2. Note that there is a constant $c$ such that

$$\pi(x) < c \frac{x}{\log x}.$$

Pick $x$ sufficiently large so that

$$\tau(x) < x^{1/2}.$$

and

$$x^{1/2} \log x < (\log x - c)x.$$

Rearranging the last inequality, we get

$$\begin{aligned} x - x^{1/2} &> c \frac{x}{\log x} \\ &= \pi(x). \end{aligned}$$

If $n = \llcorner x \lrcorner = x$ is a natural number then the number on the LHS counts the number of integers up to $n$ which are coprime to $n$ and the number on the RHS counts the number of primes up to $n$. Thus if $n$ is sufficiently large there are always numbers coprime to $n$ which are not prime.

6.10.5. (a) Suppose that $m$ and $n$ are coprime. If we have ordered factorisations $m_1, m_2, \dots, m_k$ and $n_1, n_2, \dots, n_k$ of $m$ and $n$ then $d_1, d_2, \dots, d_k$ is an ordered factorisation of $mn$, where $d_i = m_i n_i$. Conversely if $d_1, d_2, \dots, d_k$ is an ordered factorisation of $mn$ then we can write $d_i = m_i n_i$ where $m_i$ divides $m$ and $n_i$ divides $n$. It follows that $m_1, m_2, \dots, m_k$ and $n_1, n_2, \dots, n_k$ are ordered factorisations of $m$ and $n$.

Thus the number of ordered factorisations of $mn$ is equal to the number of ordered factorisations of $m$ times the number of ordered factorisations of $n$, so that

$$\tau_k(mn) = \tau_k(m)\tau_k(n),$$

and $\tau_k$ is multiplicative.

(b) If $d_1, d_2, \dots, d_k$ are integers such that

$$p^e = d_1 d_2 \dots d_k,$$

then $d_i = p^{e_i}$ is a power of $p$ and

$$e = \sum_{i=1}^{k} e_i.$$

So we just want to count the number of ways to write $e$ as a sum of $k$ ordered non-negative integers. For each $e_i$ we have at most $e + 1$ choices, so

$$\tau_k(p^e) \leq (e+1)^k.$$

More generally one can use stars and bars to get

$$\tau_k(p^e) = \binom{e+k-1}{k-1}.$$

(c) Let

$$f(n) = \frac{\tau_k(n)}{n^\delta}.$$

Then $f(n)$ is multiplicative. Thus we may assume that $n = p^e$ is a power of a prime. In this case

$$f(p^e) \leq \frac{(e+1)^k}{p^{e\delta}}.$$

As $k$ is fixed, this goes to zero as $p^e$ goes to infinity, since in this case either $e$ or $p$ goes to infinity.

6.11.1. Since

$$\sigma(m) = \sum_{d|m} d$$

we have

$$\begin{aligned}
\sum_{m=1}^{n} \sigma(m) &= \sum_{m=1}^{n} \sum_{d|m} d \\
&= \sum_{d_1 d_2 \le n} d_2 \\
&= \sum_{d_1=1}^{n} \sum_{d_2=1}^{n/d_1} d_2 \\
&= \sum_{d=1}^{n} \frac{\llcorner n/d \lrcorner^2 + \llcorner n/d \lrcorner}{2} \\
&= \frac{1}{2} \sum_{d=1}^{n} \frac{n^2}{d^2} + O\left( \sum_{d=1}^{n} \frac{n}{d} \right) \\
&= \frac{n^2}{2} \left( \sum_{d=1}^{\infty} \frac{1}{d^2} - \sum_{d=n+1}^{\infty} \frac{1}{d^2} \right) + O(n \log n) \\
&= \frac{n^2}{2} \zeta(2) + O\left( n^2 \sum_{d=n+1}^{\infty} \frac{1}{d^2} \right) + O(n \log n) \\
&= \frac{n^2 \pi^2}{6} + O(n) + O(n \log n) \\
&= \frac{n^2 \pi^2}{6} + O(n \log n).
\end{aligned}$$

6.11.2. (a) We apply partial summation to

$$\lambda_n = n \qquad c_n = \tau(n) \qquad \text{and} \qquad f(x) = \frac{1}{x}.$$

We get

$$\sum_{n \le x} \frac{\tau(n)}{n} = \frac{\sum_{n \le x} \tau(n)}{x} + \int_1^x \frac{\sum_{n \le t} \tau(n)}{t^2} \, \mathrm{d}t.$$

Now

$$\sum_{n \le x} \tau(n) = x \log x + (2\gamma - 1)x + O(x^{1/2}).$$

Thus

$$\int_1^x \frac{\sum_{n\le t}\tau(n)}{t^2}\,\mathrm{d}t = \int_1^x \frac{\log t}{t}\,\mathrm{d}t + (2\gamma-1)\int_1^x \frac{1}{t}\,\mathrm{d}t + \int_1^x \frac{O(t^{1/2})}{t^2}\,\mathrm{d}t$$
$$= \left[\frac{1}{2}\log^2 t\right]_1^x + (2\gamma-1)\Big[\log t\Big]_1^x + O\left(\int_1^\infty \frac{1}{t^{3/2}}\,\mathrm{d}t\right)$$
$$= \frac{1}{2}\log^2 x + (2\gamma-1)\log x + O(1).$$

Hence

$$\sum_{n\le x}\frac{\tau(n)}{n} = \frac{1}{2}\log^2 x + 2\gamma\log x + O(1).$$

(b) We apply partial summation to

$$\lambda_n = n \qquad c_n = \tau(n) \qquad \text{and} \qquad f(x) = \frac{1}{\log x}.$$

We get

$$\sum_{n\le x}\frac{\tau(n)}{\log n} = \frac{\sum_{n\le x}\tau(n)}{\log x} + \int_1^x \frac{\sum_{n\le t}\tau(n)}{t\log^2 t}\,\mathrm{d}t.$$

Now

$$\sum_{n\le x}\tau(n) = x\log x + (2\gamma-1)x + O(x^{1/2}).$$

We have

$$\int_1^x \frac{\sum_{n\le t}\tau(n)}{t\log^2 t}\,\mathrm{d}t = \int_1^x \frac{1}{\log t}\,\mathrm{d}t + (2\gamma-1)\int_1^x \frac{1}{\log^2 t}\mathrm{d}t + \int_1^x \frac{O(t^{1/2})}{t\log^2 t}\,\mathrm{d}t.$$

Now we have to estimate all three integrals. The first differs by a constant from the logarithmic integral and we know

$$\mathrm{li}(x) = \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right).$$

For the second integral we have

$$\int_1^x \frac{1}{\log^2 t}\mathrm{d}t = \int_1^{\sqrt{x}} \frac{1}{\log^2 t}\mathrm{d}t + \int_{\sqrt{x}}^x \frac{1}{\log^2 t}\mathrm{d}t$$
$$= O(\sqrt{x}) + O\left(\frac{x}{\log^2 x}\right)$$
$$= O\left(\frac{x}{\log^2 x}\right).$$

For the third integral we have

$$\left|\int_1^x \frac{O(t^{1/2})}{t\log^2 t}\,\mathrm{d}t\right| \le O\left(\int_1^x \frac{1}{t^{1/2}\log^2 t}\,\mathrm{d}t\right)$$
$$= O\left(\frac{x}{\log^2 x}\right).$$

Hence

$$\sum_{n\le x}\frac{\tau(n)}{n} = x + 2\gamma\frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right).$$

6.11.4. We apply partial summation to

$$\lambda_n = n \qquad c_n = \varphi(n) \qquad \text{and} \qquad f(x) = \frac{1}{x}.$$

We get

$$\sum_{n\le x}\frac{\varphi(n)}{n} = \frac{\sum_{n\le x}\varphi(n)}{x} + \int_1^x \frac{\sum_{n\le t}\varphi(n)}{t^2}\,\mathrm{d}t.$$

Now

$$\sum_{n\le x}\varphi(n) = \frac{3x^2}{\pi^2} + O(x\log x).$$

Thus

$$\int_1^x \frac{\sum_{n\le t}\varphi(n)}{t^2}\,\mathrm{d}t = \int_1^x \frac{3}{\pi^2}\,\mathrm{d}t + \int_1^x \frac{O(t\log t)}{t^2}\,\mathrm{d}t$$
$$= \frac{3x}{\pi^2} + O\left(\int_1^x \frac{\log t}{t}\,\mathrm{d}t\right)$$
$$= \frac{3x}{\pi^2} + O(\log x).$$

Hence

$$\sum_{n\le x}\frac{\varphi(n)}{n} = \frac{6x}{\pi^2} + O(\log x).$$

It follows that

$$\sum_{n\le x}\frac{\varphi(n)}{n} \sim \frac{x}{\zeta(2)} > \frac{x}{2}.$$

In particular the numbers $\varphi(n)/n$ are not uniformly distributed in the interval $[0,1]$, since the limit of the average is

$$\frac{1}{\zeta(2)}$$

and not $1/2$.

6.11.5. (a) We want to check that

$$\frac{1}{\varphi(n)} = \frac{1}{n}\sum_{d|n}\frac{\mu^2(d)}{\varphi(d)}.$$

Since both sides are multiplicative we may assume that $n = p^e$ is a power of a prime. In this case

$$\begin{aligned}
\frac{1}{n}\sum_{d|n}\frac{\mu^2(d)}{\varphi(d)} &= \frac{1}{p^e}\sum_{i=0}^{e}\frac{\mu^2(p^i)}{\varphi(p^i)}\\
&= \frac{1}{p^e}\left(\frac{1}{\varphi(1)} + \frac{1}{\varphi(p)}\right)\\
&= \frac{1}{p^e}\left(1 + \frac{1}{(p-1)}\right)\\
&= \frac{1}{p^e}\left(\frac{p}{p-1}\right)\\
&= \frac{1}{p^e}\left(\frac{1}{1-1/p}\right)\\
&= \frac{1}{\varphi(p^e)}.
\end{aligned}$$

(b) We have

$$\begin{aligned}
\sum_{n\le x}\frac{1}{\varphi(n)} &= \sum_{n\le x}\frac{1}{n}\sum_{d|n}\frac{\mu^2(d)}{\varphi(d)}\\
&= \sum_{d_1d_2\le x}\frac{1}{d_1d_2}\frac{\mu^2(d_2)}{\varphi(d_2)}\\
&= \sum_{d_2=1}^{x}\frac{\mu^2(d_2)}{d_2\varphi(d_2)}\sum_{d_1=1}^{x/d_2}\frac{1}{d_1}\\
&= \sum_{d=1}^{x}\frac{\mu^2(d)}{d\varphi(d)}(\log x - \log d)\\
&= \sum_{d=1}^{\infty}\frac{\mu^2(d)}{d\varphi(d)}\log x - \sum_{d=1}^{x}\frac{\log d\mu^2(d)}{d\varphi(d)} - \sum_{d>x}\frac{\mu^2(d)}{d\varphi(d)}\\
&= A\log x + O(1),
\end{aligned}$$

where we used the fact that

$$\varphi(n) \gg \frac{n}{\log\log n},$$

to conclude that the last two sums converge.