

MODEL ANSWERS TO THE SEVENTH HOMEWORK

6.8.4 (a) We know by Euler's theorem that

$$a^h \equiv 1 \pmod{k}.$$

As χ is completely multiplicative, we have

$$\begin{aligned} 1 &= \chi(a^h) \\ &= \chi(a)^h. \end{aligned}$$

Thus $\chi(a)$ is an h th root of unity.

(b) Note that

$$3^2 \equiv 1, \quad 5^2 \equiv 1 \quad \text{and} \quad 7^2 \equiv 1 \pmod{8}.$$

Thus the image of 3, 5 and 7 under a character is ± 1 . On the other hand,

$$3 \cdot 5 \equiv 7 \pmod{8}.$$

Thus if we assign ± 1 to 3 and 5 then the value of 7 is determined by multiplicativity. The four functions listed list all possible ways to assign ± 1 to 3 and 5. Thus there are at most four characters. On the other hand, it is not hard to check that these functions are completely multiplicative.

It is also not hard to check that the product of two characters is a character. Associativity is clear. χ_0 plays the role of 1 and each character is its own inverse. Thus we do have a group.

(c) Since

$$|\chi_i(a)| = 1,$$

it follows that

$$\sum_{n=1}^{\infty} \frac{\chi_i(n)}{n^s}$$

converges absolutely for $s > 1$.

By (9.3)

$$\begin{aligned}
 L(s, \chi_0) &= \prod_{p:\text{odd}} \left(1 - \frac{1}{p^s}\right)^{-1} \\
 &\quad \left(1 - \frac{1}{2^s}\right) \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \\
 &\quad \left(1 - \frac{1}{2^s}\right) \zeta(s).
 \end{aligned}$$

It follows that

$$\begin{aligned}
 \lim_{s \rightarrow 1^+} (s-1)L(s, \chi_0) &= \lim_{s \rightarrow 1^+} \left(1 - \frac{1}{2^s}\right) \lim_{s \rightarrow 1^+} \zeta(s) \\
 &= 1/2 \cdot 1 \\
 &= 1/2.
 \end{aligned}$$

Thus

$$L(s, \chi_0) \sim \frac{1}{2}(s-1)^{-1}.$$

To show that $L(s, \chi_i)$ is continuous at $s = 1$ for $i > 0$ we show uniform convergence of the series defining $L(s, \chi_i)$. Fix a closed interval $I = [s_0, s_1]$, where $s_0 > 0$. We have to show that given $\epsilon > 0$ there is an m_0 such that

$$\left| \sum_{k=m}^n \frac{\chi_i(k)}{k^s} \right| < \epsilon$$

for all $m \geq m_0$, for all $s \in I$.

The idea is to apply Abel's partial summation formula. Note first that if $i > 0$ then

$$\left| \sum_{k=m}^n \chi_i(k) \right| \leq 2,$$

regardless of m and n . If we put $A_{m-1} = 0$ and

$$A_k = \sum_{l=m}^k \chi_i(l)$$

for $k \geq m$ then

$$\begin{aligned}
\left| \sum_{k=m}^n \frac{\chi_i(k)}{k^s} \right| &= \left| \sum_{k=m}^n \frac{A_k - A_{k-1}}{k^s} \right| \\
&= \left| \sum_{k=m}^n A_k(k^{-s} - (k+1)^{-s}) + A_n(n+1)^{-s} - A_{m-1}m^{-s} \right| \\
&\leq \sum_{k=m}^n 2(k^{-s} - (k+1)^{-s}) + 2(n+1)^{-s} \\
&= 2m^{-s} \\
&\leq 2m^{-s_0}.
\end{aligned}$$

On the other hand, $m^{-s_0} < \epsilon$ for all m sufficiently large. Thus $L(s, \chi_i)$ is continuous for $s > 0$, if $i > 0$. We have

$$\begin{aligned}
L(s, \chi_1) &= \prod_{p \equiv 1,5} \left(1 - \frac{1}{p^s}\right)^{-1} \prod_{p \equiv 3,7} \left(1 + \frac{1}{p^s}\right)^{-1} \\
L(s, \chi_2) &= \prod_{p \equiv 1,3} \left(1 - \frac{1}{p^s}\right)^{-1} \prod_{p \equiv 5,7} \left(1 + \frac{1}{p^s}\right)^{-1} \\
L(s, \chi_3) &= \prod_{p \equiv 1,7} \left(1 - \frac{1}{p^s}\right)^{-1} \prod_{p \equiv 3,5} \left(1 + \frac{1}{p^s}\right)^{-1},
\end{aligned}$$

where we work modulo 8.

(d) If we multiply the three equations above together with $\zeta(a)$ we get:

$$\prod_{i=0}^3 L(s, \chi_i) = \prod_{p \equiv 1} \left(1 - \frac{1}{p^s}\right)^{-4} \prod_{p \equiv 3,5,7} \left(1 - \frac{1}{p^{2s}}\right)^{-2},$$

since if $p \equiv 1 \pmod{8}$ the factor

$$\left(1 \pm \frac{1}{p^s}\right)^{-1}$$

occurs in all expressions with a minus sign but for $p \equiv 3, 5, 7 \pmod{8}$ there are two minuses and two pluses.

(e) We want to show that

$$\frac{L(s, \chi_2)L(s, \chi_3)}{L(s, \chi_0)L(s, \chi_1)} = \prod_{\substack{p \equiv 5 \\ \pmod{8}}} \frac{(1 - p^{-s})^4}{(1 - p^{-2s})^2}.$$

Both sides are the products of factors of the form

$$\left(1 - \frac{1}{p^s}\right) \quad \text{and} \quad \left(1 + \frac{1}{p^s}\right)$$

to various powers. It is enough to check we get the same powers on both sides.

We check that if p appears on the LHS then $p \equiv 5 \pmod{8}$.

Suppose that $p \equiv 1 \pmod{8}$. Then

$$\left(1 - \frac{1}{p^s}\right)^{-1}$$

occurs twice on the top and bottom of the LHS, so not at all in total.

If $p \equiv 3 \pmod{8}$ then

$$\left(1 - \frac{1}{p^s}\right)^{-1}$$

occurs once, as a factor of $L(s, \chi_2)$, on the top and once on the bottom, as a factor of $L(s, \chi_0)$. Similarly

$$\left(1 + \frac{1}{p^s}\right)^{-1}$$

occurs once, as a factor of $L(s, \chi_3)$, on the top and once on the bottom, as a factor of $L(s, \chi_1)$. In total there is no contribution from $p \equiv 3 \pmod{8}$. One can check that something similar happens for $p \equiv 7 \pmod{8}$, with the roles of $L(s, \chi_2)$ and $L(s, \chi_0)$ switched with $L(s, \chi_3)$ and $L(s, \chi_1)$.

Now suppose that $p \equiv 5 \pmod{8}$. Then

$$\left(1 - \frac{1}{p^s}\right)^{-1}$$

occurs twice on the bottom of the LHS. Thus we get a factor of

$$\left(1 - \frac{1}{p^s}\right)^2$$

On the other hand

$$\left(1 + \frac{1}{p^s}\right)^{-1}$$

occurs twice on the top. Thus we get a factor of

$$\left(1 + \frac{1}{p^s}\right)^{-2}.$$

Thus we get the same powers as on the RHS.

By symmetry we get

$$\frac{L(s, \chi_1)L(s, \chi_3)}{L(s, \chi_0)L(s, \chi_2)} = \prod_{p \equiv 3 \pmod 8} \frac{(1 - p^{-s})^4}{(1 - p^{-2s})^2}$$

$$\frac{L(s, \chi_1)L(s, \chi_2)}{L(s, \chi_0)L(s, \chi_3)} = \prod_{p \equiv 7 \pmod 8} \frac{(1 - p^{-s})^4}{(1 - p^{-2s})^2}.$$

(f) By (e) we have

$$(1 - p^{-2s})^{-2} \frac{L(s, \chi_0)L(s, \chi_1)}{L(s, \chi_2)L(s, \chi_3)} = \prod_{p \equiv 5 \pmod 8} (1 - p^{-s})^{-4}$$

Taking the limit as s approaches 1 from above we see that

$$\lim_{s \rightarrow 1^+} \prod_{p \equiv 5 \pmod 8} (1 - p^{-s})^{-1} = \infty.$$

Arguing as in (6.8.1) this implies that

$$\sum_{p \equiv 5 \pmod 8} \frac{1}{p}$$

diverges. Similarly both

$$\sum_{p \equiv 3 \pmod 8} \frac{1}{p} \quad \text{and} \quad \sum_{p \equiv 7 \pmod 8} \frac{1}{p}$$

diverge.

6.9.1. We have

$$\begin{aligned} \frac{\pi((1 + \alpha)x) - \pi(x)}{x/\log x} &> c_1(1 + \alpha)x/\log(1 + \alpha)x \cdot \log x/x - c_2 \\ &= c_1(1 + \alpha) \log x/(\log x + \log(1 + \alpha)) - c_2 \\ &= c_1(1 + \alpha)/(1 + \log(1 + \alpha)/\log x) - c_2 \\ &= c_1(1 + \alpha)(1 - \log(1 + \alpha)/\log x + \log^2(1 + \alpha)/\log^2 x + \dots) - c_2 \\ &= c_1(1 + \alpha - \beta) + O\left(\frac{1}{\log x}\right). \end{aligned}$$

If $1 + \alpha > \beta$ then the last expression goes to infinity, so that the number of primes between x and $(1 + \alpha)x$ tends to infinity.

6.9.3. Note that every integer $7 \leq m \leq 19$ is a sum of distinct primes $p < 13$

$$7 = 7, \quad 8 = 5+3, \quad 9 = 7+2, \quad 10 = 7+3, \quad 11 = 11, \quad 12 = 7+5, \quad 13 = 11+2,$$

$$14 = 11+3, \quad 15 = 7+5+3, \quad 16 = 11+5, \quad 17 = 7+5+3+2, \quad 18 = 11+7, \quad 19 = 11+5+3.$$

Suppose that $19 < m \leq 26$. Then $7 \leq m - 13 \leq 13$. Thus $m - 13$ is a sum of primes less than 13 and adding 13, m is a sum of distinct primes. Thus every integer $7 \leq m \leq 26$ is a sum of distinct primes $p \leq 13$.

Now suppose that we know every integer $7 \leq m \leq 2p$ is a sum of distinct primes at most p . Pick a prime $p < q \leq 2p$. If $2p < m \leq 2q$ then $m - q \leq q \leq 2p$. In this case $m - q$ is a sum of distinct primes less than p and adding on q , m is a sum of distinct primes at most $q > p$.

6.9.4. (a) We apply partial summation to

$$\lambda_n = p_n \quad c_n = 1 \quad \text{and} \quad f(x) = \log x.$$

We get

$$\sum_{p \leq x} \log x = \pi(x) \log x - \int_2^x \frac{\pi(t)}{t} dt.$$

Consider estimating

$$\int_2^x \frac{dt}{\log^n t}$$

where $n = 1$ or 2 . We split this integral into two parts

$$\begin{aligned} \int_2^x \frac{dt}{\log^n t} &= \int_2^{\sqrt{x}} \frac{dt}{\log^n t} + \int_{\sqrt{x}}^x \frac{dt}{\log^n t} \\ &\leq \frac{\sqrt{x}}{\log^n t} + \frac{x}{\log^n \sqrt{x}} \\ &= O(\sqrt{x}) + O\left(\frac{x}{\log^n x}\right) \\ &= O\left(\frac{x}{\log^n x}\right). \end{aligned}$$

As

$$\pi(x) = \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right)$$

we see that

$$\begin{aligned} \int_2^x \frac{\pi(t)}{t} dt &= O\left(\frac{x}{\log x}\right) + O\left(\frac{x}{\log^2 x}\right) \\ &= O\left(\frac{x}{\log x}\right). \end{aligned}$$

Thus

$$\vartheta(x) = x + O\left(\frac{x}{\log x}\right).$$

(b) Fix $\epsilon > 0$. As

$$O\left(\frac{x}{\log x}\right) = o(x),$$

we have

$$(1 - \epsilon)x < \sum_{p \leq x} \log p < (1 + \epsilon)x$$

for x sufficiently large. If we exponentiate then we get

$$e^{(1-\epsilon)x} < \prod_{p \leq x} p < e^{(1+\epsilon)x}$$

for x sufficiently large.

(c) If we take the inequality above and divide through by e^x then we get

$$e^{-\epsilon x} < \frac{\prod_{p \leq x} p}{e^x} < e^{\epsilon x}$$

for x sufficiently large.

It does not seem possible to decide whether or not the ratio in the middle tends to one or not from this.

6.9.6. Let $m = n + k$. If $k \geq n$ then pick a prime

$$\lfloor m/2 \rfloor < p \leq 2\lfloor m/2 \rfloor.$$

If p divides two integers less than m then p divides m , in which case either $m = p$ or $m = 2p$. If $m = p$ then m is odd and

$$\begin{aligned} p &\leq 2\lfloor m/2 \rfloor \\ &< m \\ &= p, \end{aligned}$$

a contradiction. If $m = 2p$ then

$$\begin{aligned} p &= m/2 \\ &= \lfloor m/2 \rfloor \\ &< p, \end{aligned}$$

a contradiction. Thus p divides only one integer $l \leq m$. If we add

$$\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{m}$$

in the usual way then the denominator is divisible by p and the numerator is congruent to 1 modulo p , since all terms in the numerator other than the term corresponding to $1/l$ are divisible by p .

Thus if $k > n$ then

$$\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{m}.$$

is not an integer.

Now suppose that $k \leq n$. Then

$$\begin{aligned} \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{m} &\leq \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n} \\ &< \frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n} \\ &= 1. \end{aligned}$$

As the first sum is positive, it is not an integer.