MODEL ANSWERS TO THE SIXTH HOMEWORK

6.8.1. We know that

$$\lim_{s\to 1^+}\prod_q\left(1-\frac{1}{q^{-s}}\right)^{-1}=\infty.$$

If we take logs we get

$$\lim_{s\to 1^+}\sum_q\log\left(1-\frac{1}{q^{-s}}\right)=-\infty.$$

As

$$\log\left(1-\frac{1}{q^{-s}}\right)\geq\frac{-2}{q^{-s}},$$

this implies

$$\lim_{s\to 1^+}\sum_q-\frac{2}{q^{-s}}=-\infty.$$

Thus

$$\lim_{s\to 1^+}\sum_q\frac{1}{q^{-s}}=\infty.$$

This implies

$$\sum_q\frac{1}{q},$$

diverges.

6.8.2. In the course of the proof of (9.2) we established that

$$\lim_{s\to 1^+}(s-1)\zeta(s)=1.$$

We also proved that $L(s)$ is continuous at $s=1$ and that $L(1)\neq 0$. Thus

$$\begin{aligned}\lim_{s\to 1^+}(s-1)\zeta(s)L(s)&=\lim_{s\to 1^+}(s-1)\zeta(s)\cdot\lim_{s\to 1^+}L(s)\\&=L(1)\\&\neq 0.\end{aligned}$$

As we showed

$$\zeta(s)L(s)=\left(1-\frac{1}{2^s}\right)^{-1}\prod_q\left(1-\frac{1}{q^s}\right)^{-2}\cdot\prod_r\left(1-\frac{1}{r^{2s}}\right)^{-1},$$

it follows that

$$L(1) = \lim_{s\to1^+}\left(1-\frac{1}{2^s}\right)^{-1} \cdot \lim_{s\to1^+}(s-1)\prod_q\left(1-\frac{1}{q^s}\right)^{-2} \cdot \lim_{s\to1^+}\prod_r\left(1-\frac{1}{r^{2s}}\right)^{-1}.$$

We proved that the last functions is continuous and non-zero at 1. As the first function is also continuous and non-zero at 1, it follows that

$$\lim_{s\to1^+}(s-1)\prod_q\left(1-\frac{1}{q^s}\right)^{-2} = A$$

where A is non-zero.
Thus

$$\lim_{s\to1^+}(s-1)\prod_q\left(1-\frac{1}{q^s}\right)^{-1} = 0.$$

Suppose that the limit

$$\lim_{s\to1^+}\prod_r\left(1-\frac{1}{r^s}\right)^{-1} = B$$

exists and is finite. As

$$\zeta(s) = \left(1-\frac{1}{2^s}\right)^{-1}\prod_q\left(1-\frac{1}{q^s}\right)^{-1} \cdot \prod_r\left(1-\frac{1}{r^s}\right)^{-1},$$

Then

$$\begin{aligned}1 &= \lim_{s\to1^+}\left(1-\frac{1}{2^s}\right)^{-1} \lim_{s\to1^+}(s-1)\prod_q\left(1-\frac{1}{q^s}\right)^{-1} \cdot \lim_{s\to1^+}\prod_r\left(1-\frac{1}{r^s}\right)^{-1}\\ &= 2\cdot 0\cdot B\\ &= 0,\end{aligned}$$

a contradiction.
Thus

$$\lim_{s\to1^+}\prod_r\left(1-\frac{1}{r^s}\right)^{-1}$$

diverges. Arguing as in (6.8.1) it follows that the sum of the reciprocals of the primes congruent to 3 modulo 4 diverges.

6.8.3. We have

$$\prod_p \sum_{k=0}^{\infty} f(p^k) = \sum_{r,p_1,p_2,\dots,p_r,k_1,k_2,\dots,k_r} \prod_{i=1}^{r} f(p_i^{k_i})$$
$$= \sum_{r,p_1,p_2,\dots,p_r,k_1,k_2,\dots,k_r} f(\prod_{i=1}^{r} p_i^{k_i})$$
$$= \sum_{n=1}^{\infty} f(n),$$

where we are allowed to rearrange the sum by absolute convergence. If $f(n)$ is completely multiplicative and $f(p) = 1$ then $f(p^k) = 1$ for every k so that

$$\sum_{k=0}^{\infty} f(p^k)$$

diverges, a contradiction. We have

$$\prod_p (1 - f(p))^{-1} = \prod_p \sum_{k=0}^{\infty} f(p)^k$$
$$= \prod_p \sum_{k=0}^{\infty} f(p^k),$$

and so we done by the first part.

2. (a) Suppose that

$$g(x) = \frac{p(x)}{q(x)} e^{-1/x^2},$$

for $x \neq 0$, where $p(x)$ and $q(x)$ are polynomials. Then

$$g'(x) = \frac{p'(x)q(x)e^{-1/x^2} + 2p(x)q(x)e^{-1/x^2}/x^3 - q'(x)p(x)e^{-1/x^2}}{q^2(x)}$$
$$= \frac{p'(x)q(x)x^3 + 2p(x)q(x) - q'(x)p(x)x^3}{q^2(x)x^3} e^{-1/x^2}.$$

Thus we are done by induction on n.

(b) Suppose that

$$g(x) = \frac{p(x)}{q(x)} e^{-1/x^2},$$

for $x \neq 0$, where $p(x)$ and $q(x)$ are polynomials and $g(0) = 0$. We first check that

$$\lim_{x \to 0} \frac{p(x)}{q(x)} e^{-1/x^2} = 0.$$

Factoring $p(x)$ and $q(x)$ it suffices to prove that

$$\lim_{x \to 0} x^k e^{-1/x^2} = 0,$$

for any integer k. Replacing x by $1/x$ it suffices to prove that

$$\lim_{x \to \infty} x^k e^{-x^2} = 0,$$

for any integer k. This is easy (and well-known).

We now compute the derivative of $g(x)$.

$$\begin{aligned} \lim_{x \to 0} \frac{g(x) - g(0)}{x - 0} &= \lim_{x \to 0} \frac{p(x)}{q(x)} e^{-1/x^2} \\ &= 0. \end{aligned}$$

Thus

$$f^{(n)}(0) = 0.$$

by induction on n.

(c) The Taylor series for $f(x)$ has all of its coefficients zero. This surely converges everywhere and defines the zero function but this is not equal to $f(x)$.