MODEL ANSWERS TO THE FIFTH HOMEWORK

6.7.4. (a) We apply partial summation to

$$\lambda_n = p_n \qquad c_n = 1 \qquad \text{and} \qquad f(x) = \log x.$$

We get

$$\sum_{p \leq x} \log x = \pi(x) \log x - \int_2^x \frac{\pi(t)}{t} \, \mathrm{d}t.$$

Now

$$\int_2^x \frac{\mathrm{d}t}{\log t} = 0(x).$$

As

$$\pi(x) \leq c \frac{x}{\log x},$$

we see that

$$\int_2^x \frac{\pi(t)}{t} \, \mathrm{d}t = 0(x).$$

Thus

$$\vartheta(x) < c_2 x.$$

If we exponentiate then

$$\prod_{p \leq x} p < A_1^x.$$

(b) We apply partial summation to

$$\lambda_n = p_n \qquad c_n = \log p_n \qquad \text{and} \qquad f(x) = \frac{1}{\log x}.$$

We get

$$\sum_{p \leq x} 1 = \frac{\vartheta(x)}{\log x} - \int_2^x \frac{\vartheta(t)}{t \log^2 t} \, \mathrm{d}t.$$

Now

$$\begin{aligned} \int_2^x \frac{\mathrm{d}t}{\log^2 t} &= \int_2^{x^{1/2}} \frac{\mathrm{d}t}{\log^2 t} + \int_{x^{1/2}}^x \frac{\mathrm{d}t}{\log^2 t} \\ &= O\left(x^{1/2}\right) + O\left(\frac{x}{\log x}\right) \\ &= O\left(\frac{x}{\log x}\right). \end{aligned}$$

As

$$\vartheta(x) \leq c_2 x$$

we see that

$$\int_2^x \frac{\vartheta(t)}{t \log^2 t} \, \mathrm{d}t = O\left(\frac{x}{\log x}\right).$$

Thus

$$\pi(x) = \frac{\vartheta(x)}{\log x} + O\left(\frac{x}{\log x}\right).$$

6.7.5. (a) Let p be a prime. We have to check that the same multiple of $\log p$ appears on both sides. Note that the largest power of p which divides

$$[1, 2, 3, \dots, n]$$

is equal to the largest power of p which is less than n. Suppose that $p^m \leq n$ but that $p^{m+1} > n$. Thus the coefficient of $\log p$ which appears on the RHS is m.
On the LHS we also get m copies of p, since there are m integers i such that $p^i \leq n$, $1 \leq i \leq m$.
(b) Let p be a prime. We have to check that the same multiple of $\log p$ appears on both sides. But $\log p$ appears on the LHS is $p^i \leq x$ and on the RHS if $p \leq x^{1/i}$, that is, if $p^i \leq x$. Thus

$$\psi(x) = \sum_m \vartheta(x^{1/m}).$$

Note that $2 > x^{1/m}$ if and only if $2^m > x$ if and only if

$$m > \frac{\log x}{\log 2}.$$

We have

$$\begin{aligned}
\sum_{m>1} \vartheta(x^{1/m}) &= \sum_{m=2}^{\log x/\log 2} \vartheta(x^{1/m}) \\
&\leq c(x^{1/2} + \sum_{m=3}^{\log x/\log 2})x^{1/3} \\
&\leq c(x^{1/2} + x^{1/3} \log x) \\
&= O(x^{1/2}).
\end{aligned}$$

Thus

$$\begin{aligned}\psi(x) &= \sum_m \vartheta(x^{1/m}) \\ &= \vartheta(x) + O(x^{1/2}) \\ &= O(x).\end{aligned}$$

If we exponentiate then we see that

$$\begin{aligned}[1, 2, \ldots, n] &= e^{\psi(n)} \\ &= A_2^n.\end{aligned}$$

6.7.6. Since

$$\vartheta(n) - \vartheta(n-1) = \begin{cases} \log n & \text{if } n \text{ is prime} \\ 0 & \text{otherwise,} \end{cases}$$

it is easy to see that

$$\sum_{p \leq N} F(p) = \sum_{n=2}^{N} (\vartheta(n) - \vartheta(n-1)) \frac{F(n)}{\log n}.$$

We apply Abel summation formula to the RHS with

$$a_k = \vartheta(k) - \vartheta(k-1) \qquad \text{and} \qquad b_k = \frac{F(k)}{\log k}.$$

The RHS becomes

$$\sum \theta(k) \left(\frac{F(k)}{\log k} - \frac{F(k-1)}{\log k - 1} \right) + \theta(N) \frac{F(N+1)}{\log(N+1)}.$$

Since

$$\frac{F(k)}{\log k} - \frac{F(k-1)}{\log k - 1} \leq 0$$

this sum is at most

$$\sum c_1 k \left(\frac{F(k)}{\log k} - \frac{F(k-1)}{\log k - 1} \right) + c_2 N \frac{F(N+1)}{\log(N+1)}.$$

where

$$c_1 k \leq \vartheta(k) \leq c_2 k.$$

If we reverse the partial summation then we get

$$\sum c_1 \frac{F(k)}{\log k} + (c_2 - c_1) N \frac{F(N+1)}{\log(N+1)}.$$

If

$$\sum c_1 \frac{F(k)}{\log k}$$

converges then

$$\frac{F(k)}{k} = o(k)$$

so that

$$c_1 \sum \frac{F(k)}{\log k} + (c_2 - c_1) N \frac{F(N+1)}{\log(N+1)}.$$

converges. Thus

$$\sum_{p \leq N} F(p)$$

converges. On the other hand

$$\sum \theta(k) \left(\frac{F(k)}{\log k} - \frac{F(k-1)}{\log k - 1} \right) + \theta(N) \frac{F(N+1)}{\log(N+1)}.$$

is at least

$$\sum c_2 k \left(\frac{F(k)}{\log k} - \frac{F(k-1)}{\log k - 1} \right) + c_1 N \frac{F(N+1)}{\log(N+1)}.$$

This is equal to

$$\sum c_2 \frac{F(k)}{\log k} + (c_1 - c_2) N \frac{F(N+1)}{\log(N+1)}.$$

If

$$\sum_{p \leq N} F(p)$$

converges then

$$\sum c_2 \frac{F(k)}{\log k} + (c_1 - c_2) N \frac{F(N+1)}{\log(N+1)}$$

converges. As

$$\pi(N) \geq c_3 \frac{N}{\log N},$$

it follows that

$$N \frac{F(N+1)}{\log(N+1)}$$

goes to zero. But then

$$\sum \frac{F(k)}{\log k}$$

converges.

6.7.7. (a) We have

$$
\begin{aligned}
|\{\, n \mid p_n \le x, d_n > c\log x \,\}| \cdot c\log x &= c\log x \cdot \sum_{p_n \le x, d_n > c\log x} 1 \\
&= \sum_{p_n \le x, d_n > c\log x} c\log x \\
&< \sum_{p_n \le x, d_n > c\log x} d_n \\
&= p_n - p_m \\
&\le x.
\end{aligned}
$$

(b) We have

$$
\begin{aligned}
|\{\, n \mid p_n \le x, d_n > c\log p_n \,\}| &= |\{\, n \mid \sqrt{x} < p_n \le x, d_n > c\log p_n \,\}| + |\{\, n \mid p_n \le \sqrt{x}, d_n > c\log p_n \,\}| \\
&< |\{\, n \mid \sqrt{x} < p_n \le x, d_n > c\log p_n \,\}| + \sqrt{x} \\
&< |\{\, n \mid \sqrt{x} < p_n \le x, d_n > c\log \sqrt{x} \,\}| + O(\sqrt{x}) \\
&= |\{\, n \mid p_n \le x, d_n > c/2\log x \,\}| + O(\sqrt{x}) \\
&\le \frac{2x}{c\log x} + O(\sqrt{x}).
\end{aligned}
$$

We have

$$|\{\, n \mid p_n \le x, d_n \le c\log p_n \,\}| \ge x - \frac{2x}{c\log x} + O(\sqrt{x}).$$

As

$$\pi(x) \le c_2 \frac{x}{\log x},$$

it follows that if c is large enough then

$$|\{\, n \mid p_n \le x, d_n < c\log p_n \,\}| \gg \pi(x).$$