MODEL ANSWERS TO THE FOURTH HOMEWORK

6.5.2. (a) We will prove a slightly stronger result, we will allow x to be any positive real number and we will prove that

$$\varphi(x,n) = x\varphi(n) + O(\tau(n)\ulcorner x \urcorner).$$

We proceed by induction on $r = \omega(n)$. We may assume that $r > 0$. Pick a prime p dividing n and let $n = p^e m$, where m is coprime to p. If k is not coprime to n then either p divides k or k is not coprime to m. By induction

$$\begin{aligned}\varphi(x,m) &= x\varphi(m) + O(\tau(m)\llcorner x \lrcorner)\\ &= x\varphi(m) + O(\tau(n)\llcorner x \lrcorner),\end{aligned}$$

since $\omega(m) = r - 1$.
The number of integers up to xn divisible by p are

$$\llcorner \frac{xn}{p} \lrcorner$$

and so the number of integers up to xn not divisible by p is

$$\begin{aligned}\llcorner xn \lrcorner - \llcorner \frac{xn}{p} \lrcorner &= xn - \frac{xn}{p} + \{\, xn \,\} - \{\, \frac{xn}{p} \,\}\\ &= x(n - \frac{n}{p}) + \{\, xn \,\} - \{\, \frac{xn}{p} \,\}\\ &= x\varphi(n) + E,\end{aligned}$$

where $|E|$ is at most one.
Now the number of integers up to xn which are divisible by p and are not coprime to m is equal to

$$\begin{aligned}\llcorner xn/p \lrcorner - \varphi(x, n/p) &= \llcorner xn/p \lrcorner - x\varphi(n/p) + O(\tau(n/p)\llcorner x \lrcorner)\\ &= xn/p - x\varphi(n/p) + O(\tau(n)\llcorner x \lrcorner).\end{aligned}$$

Putting all of this together we are done by induction.
(b) We just have to show

$$\frac{\tau(n)}{\varphi(n)}$$

goes to zero as n goes to infinity. Note that both top and bottom are multiplicative and so the ratio is multiplicative.

If p^e is a power of the prime p then

$$\frac{\tau(p^e)}{\varphi(p^e)} = \frac{e+1}{p(p^{e-1}-1)}.$$

Suppose this ratio is bigger than one. Then $e+1 > p$ and it is easy to check that the only possibility is that $p = 2$ and $e = 2$, in which case the ratio is $3/2$.
Now if n goes to infinity, one of two things must happen. Either n has a prime factor p which goes to infinity or the exponent e of some fixed prime p goes to infinity.
Thus we are reduced to the case that $n = p^e$ is a power of a prime. If either p or e goes to infinity then it is clear that the ratio

$$\frac{e+1}{p(p^{e-1}-1)}$$

goes to zero.
6.5.3. We use the notation

$$\langle x \rangle = \llcorner x + \frac{1}{2} \lrcorner$$

to denote the nearest integer to x.
We first check that

$$\llcorner x \lrcorner = \llcorner x/2 \lrcorner + \langle x/2 \rangle.$$

Suppose that $\{\, x/2 \,\} < 1/2$. Then

$$\langle x/2 \rangle = \llcorner x/2 \lrcorner \qquad \text{and} \qquad \llcorner x \lrcorner = 2\llcorner x/2 \lrcorner$$

and so

$$\begin{aligned} \llcorner x \lrcorner &= 2\llcorner x/2 \lrcorner \\ &= \llcorner x/2 \lrcorner + \langle x/2 \rangle. \end{aligned}$$

Now supppose that $\{\, x/2 \,\} \geq 1/2$. Then

$$\langle x/2 \rangle = \llcorner x/2 \lrcorner + 1 \qquad \text{and} \qquad \llcorner x \lrcorner = 2\llcorner x/2 \lrcorner + 1$$

and so

$$\begin{aligned} \llcorner x \lrcorner &= 2\llcorner x/2 \lrcorner + 1 \\ &= \llcorner x/2 \lrcorner + \langle x/2 \rangle. \end{aligned}$$

We compute $P(x, m)$ by inclusion-exclusion

$$\begin{aligned}
P(x, m) &= A(x, m+1) + 1 \\
&= \llcorner x \lrcorner - \llcorner x/2 \lrcorner - (\sum \llcorner x/p_i \lrcorner - \sum \llcorner x/2p_i \lrcorner) + (\sum \llcorner x/p_i p_j \lrcorner - \sum \llcorner x/2p_i p_j \lrcorner) + \dots \\
&= \langle x/2 \rangle - \sum \langle x/p_i \rangle + \sum \langle x/p_i p_j \rangle + \dots \\
&= \sum_a \langle x/2a \rangle - \sum_b \langle x/2b \rangle.
\end{aligned}$$

Now let $r = \pi(\sqrt{x})$. Then

$$\begin{aligned}
\pi(x) &= r + A(x, r) \\
&= \pi(\sqrt{x}) + P(x, \pi(\sqrt{x}) - 1.
\end{aligned}$$

Note that

$$\begin{aligned}
\sqrt{200} &= \sqrt{2 \cdot 100} \\
&= \sqrt{2}\sqrt{100} \\
&\approx 14.142.
\end{aligned}$$

Now the odd primes up to 14 are 3, 5, 7, 11 and 13. Thus

$$\pi(\sqrt{200}) = 6.$$

On the other hand, one can compute

$$\begin{aligned}
P(200, 5) &= 100 - (33 + 20 + 14 + 9 + 8) + (7 + 5 + 3 + 3 + 3 + 2 + 2 + 1 + 1 + 1) - (1 + 1 + 1 \\
&= 41.
\end{aligned}$$

Thus

$$\begin{aligned}
\pi(200) &= \pi(14) + P(200, 5) - 1 \\
&= 6 + 41 - 1 \\
&= 46.
\end{aligned}$$

6.6.1. (a) We apply partial summation to

$$\lambda_n = p_n \qquad c_n = 1 \qquad \text{and} \qquad f(x) = \frac{1}{x}.$$

We get

$$\sum_{p \leq x} \frac{1}{p} = \frac{\pi(x)}{x} - \int_2^x -\frac{\pi(t)}{t^2}\, \mathrm{d}t.$$

As

$$\pi(x) = O\left(\frac{x}{\log \log x}\right)$$

the first expression on the RHS is certainly $o(1)$. Rearranging we get

$$\int_2^x \frac{\pi(t)}{t^2}\,\mathrm{d}t = \sum_{p\le x}\frac{1}{p} + o(1) \sim \log\log x.$$

(b) Suppose that $\rho(x) \ge 1+\delta \ge 1$. Then

$$\begin{aligned}\log\log x &\sim \int_2^x \frac{\pi(t)}{t^2}\,\mathrm{d}t\\ &> \int_2^x \frac{(1+\delta)t}{t^2\log t}\,\mathrm{d}t\\ &= (1+\delta)\int_2^x \frac{\mathrm{d}t}{t\log t}\\ &= (1+\delta)\Big[\log\log t\Big]_2^x\\ &> (1+\delta)\log\log x,\end{aligned}$$

and so $\delta = 0$.
Suppose that $\rho(x) \le 1-\delta \le 1$. Then

$$\begin{aligned}\log\log x &\sim \int_2^x \frac{\pi(t)}{t^2}\,\mathrm{d}t\\ &< \int_2^x \frac{(1-\delta)t}{t^2\log t}\,\mathrm{d}t\\ &= (1-\delta)\int_2^x \frac{\mathrm{d}t}{t\log t}\\ &= (1-\delta)\Big[\log\log t\Big]_2^x\\ &= (1-\delta)(\log\log x - \log\log 2),\end{aligned}$$

so that $\delta = 0$.
(c) We apply partial summation to

$$\lambda_n = p_n \qquad c_n = 1 \qquad \text{and} \qquad f(x) = \frac{\log x}{x}.$$

We get

$$\sum_{p\le x}\frac{\log p}{p} = \rho(x) + \int_2^x \pi(t)\frac{\log t - 1}{t^2}\,\mathrm{d}t.$$

Thus

$$\log(x) + O\left(\frac{\log x}{\log\log x}\right) + \rho(x) = \int_2^x \pi(t)\frac{\log t - 1}{t^2}\,\mathrm{d}t$$

Note that

$$\rho(x) = O\left(\frac{\log x}{\log\log x}\right).$$

If $\rho(t) \geq 1 + \delta \geq 1$ then the last integral is at least

$$(1+\delta)\int_2^x \frac{\mathrm{d}t}{t} + O(\log\log x) = (1+\delta)(\log x) + O(\log\log x).$$

Thus $\delta \leq 0$. Similarly, if $\rho(t) \leq 1 - \delta \leq 1$ then the last integral is at most

$$(1-\delta)\int_2^x \frac{\mathrm{d}t}{t} = (1+\delta)(\log x) + O(\log\log x).$$

Thus $\delta = 0$.

6.6.3. We have

$$\begin{aligned} x &\sim \sum_{p\leq x} \log p \\ &\leq \sum_{p\leq x} \log x \\ &= \log x \sum_{p\leq x} 1 \\ &= \pi(x)\log x. \end{aligned}$$

On the other hand

$$\begin{aligned} x &\sim \sum_{x^{1-\epsilon}\leq p\leq x} \log p \\ &\geq \sum_{x^{1-\epsilon}\leq p\leq x} \log x^{1-\epsilon} \\ &= (1-\epsilon)\log x \sum_{x^{1-\epsilon}\leq p\leq x} 1 \\ &= (1-\epsilon)(\pi(x) - \pi(x^{1-\epsilon}))\log x \\ &= (1-\epsilon)(\pi(x) + O(x^{1-\epsilon}))\log x. \end{aligned}$$

Putting these together, we see that

$$\pi(x) \sim \frac{x}{\log x}.$$

6.6.4. We apply partial summation to

$$\lambda_n = p_n \qquad c_n = \frac{1}{p_n} \qquad \text{and} \qquad f(x) = \frac{1}{\log x}.$$

Then

$$\sum_{p\le x}\frac{1}{p\log p}=\frac{\sum_{p\le x}\frac{1}{p}}{\log x}+\int_2^x\frac{\sum_{p\le t}\frac{1}{p}}{t\log^2 t}\,\mathrm{d}t$$
$$=O\left(\frac{\log\log x}{\log x}\right)+\int_2^x\frac{\sum_{p\le t}\frac{1}{p}}{t\log^2 t}\,\mathrm{d}t.$$

Note that the last integral converges, since

$$\sum_{p\le t}\frac{1}{p}\sim\log\log t$$

and the integral of

$$\frac{\log\log t}{t\log^2 t}$$

converges.

6.6.6. (a) We have

$$\begin{aligned}\sum_{k=m}^{n}A_k(b_k-b_{k-1})+A_nb_{n+1}-A_{m-1}b_m&=\sum_{k=m}^{n}A_kb_k-\sum_{k=m}^{n}A_kb_{k-1}+A_nb_{n+1}-A_{m-1}b_m\\&=\sum_{k=m}^{n}A_kb_k-\sum_{k=m+1}^{n+1}A_{k-1}b_k+A_nb_{n+1}-A_{m-1}b_m\\&=\sum_{k=m}^{n}A_kb_k-\sum_{k=m}^{n}A_{k-1}b_k\\&=\sum_{k=m}^{n}(A_k-A_{k-1})b_k\\&=\sum_{k=m}^{n}a_kb_k.\end{aligned}$$

(b) Suppose that $\sum a_n$ converges and $b_1, b_2, \ldots$ is monotonic and bounded. Possibly passing to the tail of the sequence $b_1, b_2, \ldots$ we may assume that every element of $b_1, b_2, \ldots$ has the same sign. Possibly replacing b_i by $-b_i$ we may assume that $b_i \ge 0$. Suppose that b is an upper bound for $b_1, b_2, \ldots$. Then

$$|\sum_{k=m}^{n}a_kb_k|\le b|\sum_{k=m}^{n}a_k|$$

so that the partial sums of $\sum a_kb_k$ go to zero as m goes to infinity. Thus $\sum a_kb_k$ converges.

Now suppose that $\sum a_n$ has bounded partial sums and $b_1, b_2, \ldots$ tends monotonically to zero. Then A_nb_{n+1} tends to zero so that $A_nb_{n+1} -$

$A_{m-1}b_m$ converges to zero as m and n tend to infinity. Thus to show that the partial sums of $\sum a_k b_k$ go to zero as we increase m, it suffices to prove that the partial sums of $\sum A_k(b_k - b_{k-1})$ go to zero. Possibly replacing b_k by $-b_k$ we may assume that $b_1, b_2, \ldots$ is monotonic decreasing. Let $\alpha_k = b_k - b_{k-1} \geq 0$, $\beta_k^+ = \limsup_{l\geq k} A_l$ and $\beta_k^- = \liminf_{l\geq k} A_l$.
Then $\sum \alpha_k$ is absolutely convergent and $\beta_k^\pm$ is bounded and monotonic. Thus $\sum \alpha_k \beta_k^\pm$ converges by what we already proved.
On the other hand, the partial sums of $\sum A_k(b_k - b_{k-1})$ are bounded from above by the partial sums of $\sum \alpha_k \beta_k^+$ and from below by $\sum \alpha_k \beta_k^-$.
(c) If we put $a_k = (-1)^k$ and $b_k = k^{-s}$ then

$$\left|\sum_{k=m}^{n} a_k\right| \leq 1$$

so that $\sum a_k$ has bounded partial sums and $b_1, b_2, \ldots$ tends monotonically to zero. Thus

$$\sum_k (-1)^k k^{-s} = \sum a_k b_k$$

converges by (b).
(d) Let $a_k = (k/p)$ and $b_k = k^{-s}$. Note that

$$\sum_{k=0}^{p-1} (k/p) = 0,$$

since half of the numbers between 1 and $p-1$ are squares. Thus $\sum a_k$ has bounded partial sums. As $b_1, b_2, \ldots$ tends monotonically to zero, it follows that

$$\sum_k (l/p) k^{-s} = \sum a_k b_k$$

converges by (b).
(e) Let $c_k = a_k k^{-s_0}$ and $d_k = k^{s_0 - s}$. Then $\sum c_k$ converges and $d_1, d_2, \ldots$ is monotonic and bounded. Thus

$$\sum a_k k^{-s} = \sum c_k d_k$$

converges for all $s \geq s_0$.
Similarly if we $c_k = a_k k^{-s_0}$ and $d_k = k^{s_0 - s} \log k$. Then $\sum c_k$ converges and $d_1, d_2, \ldots$ is eventually monotonic and bounded for $s > s_0$. Thus

$$\sum a_k (\log k) k^{-s} = \sum c_k d_k$$

converges for all $s > s_0$.

6.6.7. Note that if we integrate by parts then we get

$$\begin{aligned}\int_0^x \frac{\mathrm{d}t}{\log^n t} &= \int_0^x 1\cdot\frac{\mathrm{d}t}{\log^n t}\\ &= \left[\frac{t}{\log^n t}\right]_0^x + n\int_0^x \frac{t\,\mathrm{d}t}{t\log^{n+1} t}\\ &= \frac{x}{\log^n x} + n\int_0^x \frac{\mathrm{d}t}{\log^{n+1} t}.\end{aligned}$$

It follows by induction that

$$\mathrm{li}(x) = \frac{x}{\log x} + \frac{1!x}{\log^2 x} + \frac{2!x}{\log^3 x} + \cdots + \frac{(n-1)!x}{\log^n x} + n!\int_0^x \frac{\mathrm{d}t}{\log^{n+1} t}.$$

Now to estimate the last integral, we break it into three parts.

$$\int_0^x \frac{\mathrm{d}t}{\log^{n+1} t} = \int_0^2 \frac{\mathrm{d}t}{\log^{n+1} t} + \int_2^{\sqrt{x}} \frac{\mathrm{d}t}{\log^{n+1} t} + \int_{\sqrt{x}}^x \frac{\mathrm{d}t}{\log^{n+1} t}.$$

The first integral is constant. The second is over an interval of length bounded by $\sqrt{x}$ of a function bounded by a constant ($\frac{1}{\log^{n+1} 2}$) and so the second integral is $O(\sqrt{x})$. The third integral is over an interval of length bounded by x of a function which is bounded by

$$\frac{1}{\log^{n+1}\sqrt{x}} = O\left(\frac{1}{\log^{n+1} x}\right).$$

Thus the last integral is

$$O\left(\frac{x}{\log^{n+1} x}\right).$$

Therefore

$$\left|n!\int_0^x \frac{\mathrm{d}t}{\log^{n+1} t}\right| = O\left(\frac{x}{\log^{n+1} x}\right).$$

and so the result follows.
6.7.1. We have

$$\frac{-1/p}{1-1/p} = -\frac{1}{p-1} \leq \log\left(1-\frac{1}{p}\right) \leq -\frac{1}{p}.$$

Note the difference between the upper and lower bound is $1/p(p-1)$. Thus

$$\begin{aligned}\sum_{p\leq x}\log\left(1-\frac{1}{p}\right) &= -\sum_{p}\frac{1}{p}+\frac{a_p}{p(p-1)}\\ &= -\log\log x + C + O\left(\frac{1}{\log x}\right)+\sum\frac{a_p}{p(p-1)}\\ &= -\log\log x + B + O\left(\frac{1}{\log x}\right),\end{aligned}$$

where $a_p \in (0,1]$ so that the sum is convergent.
Now suppose that $f(x)$ tends to zero as x tends to infinity. We apply Cauchy's mean value theorem to $e^{f(x)}$ and $f(x)$ over the interval $[x, x_1]$. It follows that we can find $a \in (x, x_1)$ such that

$$e^{f(a)} = \frac{e^{f(x_1)} - e^{f(x)}}{f(x_1) - f(x)}.$$

If we let x_1 go to infinity, we get

$$e^{f(a)} = \frac{1 - e^{f(x)}}{-f(x)},$$

so that

$$e^{f(x)} = 1 + f(x)e^{f(a)}.$$

As $a \geq x$, and $f(x)$ is monotonic decreasing, it follows that $e^{f(a)}$ is bounded from above. Thus

$$e^{f(x)} = 1 + O(f(x)).$$

Let

$$f(x) = \sum_{p}\log\left(1-\frac{1}{p}\right) + \log\log x - B.$$

If we exponentiate then we get

$$e^{f(x)} = 1 + O\left(\frac{1}{\log x}\right).$$

Thus, taking two terms over to the other side, we get

$$\prod_{p\leq x}\left(1-\frac{1}{p}\right) = \frac{e^{-B}}{\log x} + O\left(\frac{1}{\log^2 x}\right).$$

6.7.2. (a) Suppose not. Then we may find $\epsilon > 0$ and n_0 such that if $n \geq n_0$ then $p_{n+1} \geq (1+\epsilon)p_n$. By induction it follows that

$$p_n \geq (1+\epsilon)^{n-n_0}p_{n_0}.$$

Thus

$$\log p_n \geq (n - n_0) \log(1 + \epsilon) + O(1).$$

On the other hand,

$$p_n < c_4 n \log n,$$

so that

$$\log p_n < \log n + \log \log n + O(1),$$

a contradiction.

(b) By Theorem 8.2 we may find a such that

$$\left| \sum_{p \leq x} \frac{\log p}{p} - \log x \right| < a$$

for $x > 2$. Let $c = e^{2a}$. Suppose that there is no prime between x and cx. Then

$$\begin{aligned} \left| \sum_{p \leq cx} \frac{\log p}{p} - \log cx \right| &= \left| \sum_{p \leq x} \frac{\log p}{p} - \log x - 2a \right| \\ &> |2a - a| \\ &= a, \end{aligned}$$

a contradiction.