MODEL ANSWERS TO THE SECOND HOMEWORK

6.1.10. If $x \in S$ then let $\chi_x \colon S \longrightarrow \mathbb{C}$ be the function

$$\chi_x(y) = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{otherwise.} \end{cases}$$

Note that

$$f = \sum_{x \in X} f(x)\chi_x.$$

Indeed both sides are functions $S \longrightarrow \mathbb{C}$ and it is easy to see they have the same effect on every element of S.

Thus we are reduced to the case $f = \chi_x$. In this case, the fact that both sides are equal is shown in the proof of inclusion-exclusion.

Now suppose that S consists of N real numbers. Note that if $i < j$ then

$$S_i \cap S_j = S_i \qquad \text{so that} \qquad \cap_{j=1}^{k} S_{i_j} = S_m,$$

where m is the smallest index. Note also that

$$\begin{aligned}
\sum_{s \in S_i} f(s) &= \sum_{l=1}^{i} f(x_l) \\
&= f(x_1) + f(x_2) + \cdots + f(x_i) \\
&= x_1 + (x_2 - x_1) + \cdots + (x_i - x_{i-1}) \\
&= x_i,
\end{aligned}$$

the largest element of S_i.

The union of the sets S_i is the whole of S, so the LHS is zero. The first sum on the RHS is X_N. If one brings this over to the LHS then this gives a proof of the formula (2.4.4.a).

6.2.3. We check that both sides of this equation are additive. If x and y are any positive reals then

$$\log xy = \log x + \log y,$$

so that the LHS is certainly additive. For the RHS, note that the only non-zero terms of the sum

$$\sum_{d|n} \Lambda(d)$$

are when d is a power of a prime. If m and n are coprime and d divides mn then we can write $d = d_1 d_2$ where d_1 divides m and d_2 divides n.

If d is a pure power of a prime then one of d_1 and d_2 is one and the other is equal to d. It follows that the RHS is additive as well. Thus we may assume that $n = p^e$ is a power of a prime p. In this case

$$\begin{aligned}\sum_{d|n} \Lambda(d) &= \sum_{i=0}^{e} \Lambda(p^i) \\ &= \sum_{i=1}^{e} \log p \\ &= e \log p \\ &= \log p^e \\ &= \log n.\end{aligned}$$

By Möbius inversion, we have

$$\begin{aligned}\Lambda(n) &= \sum_{d|n} \mu(d) \log(n/d) \\ &= \sum_{d|n} \mu(d)(\log n - \log d) \\ &= \log n \sum_{d|n} \mu(d) - \sum_{d|n} \mu(d) \log d \\ &= \log n M(n) - \sum_{d|n} \mu(d) \log d \\ &= -\sum_{d|n} \mu(d) \log d\end{aligned}$$

Thus

$$\sum_{d/n} \mu(d) \log d = -\Lambda(n).$$

6.2.4. Note that $\mu(n)$ is multiplicative and the product of multiplicative functions is multiplicative, so that the LHS is multiplicative.

On the other hand, if m and n are coprime then the prime factors of mn are simply the prime factors of m and n. It is easy to see that the RHS is also multiplicative.

Thus we are reduced to the case when $n = p^e$ is the power of a prime p. In this case

$$\begin{aligned}\sum_{d|n} \mu(d)f(d) &= \sum_{i=0}^{e} \mu(p^i)f(p^i) \\ &= \mu(1)f(1) + \mu(p)f(p) + 0 + \cdots + 0 \\ &= 1 - f(p).\end{aligned}$$

6.2.7. We have

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n^s} \cdot \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} &= \sum_{m=1}^{\infty} \frac{1}{m^s} \cdot \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \\ &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^s} \frac{\mu(n)}{n^s} \\ &= \sum_{m=1}^{\infty} \sum_{d=1}^{\infty} \frac{\mu(d)}{(md)^s} \\ &= \sum_{n=1}^{\infty} \sum_{d|n} \frac{\mu(d)}{n^s} \\ &= \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{d|n} \mu(d) \\ &= \sum_{n=1}^{\infty} \frac{1}{n^s} M(n) \\ &= 1,\end{aligned}$$

where we used the fact that both

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

are absolutely convergent, so that we can rearrange the terms of the sums.

6.2.8. (a) Let I be the set of natural numbers from one to n and let I^k be the Cartesian product of I with itself k times, so that I^k has cardinality n^k. Let

$$C_d = \{\, (a_1, a_2, \dots, a_k) \in I^k \,|\, \text{the greatest common divisor of } a_1, a_2, \dots, a_k \text{ and } n \text{ is } d \,\}.$$

Note that C_d is a partition of I^k indexed by the divisors of n. If $(a_1, a_2, \dots, a_k) \in C_d$ then a_i is divisible by d so that $a_i = b_i d$. In this case $(b_1, b_2, \ldots, b_k) \in J_k(n/d)$, so that

$$|C_{n/d}| = J_k(d).$$

Thus

$$\begin{aligned} n^k &= |I^k| \\ &= \sum_{d|n} |C_{n/d}| \\ &= \sum_{d|n} J_k(d). \end{aligned}$$

(b) As $F(n) = n^k$ is multiplicative, it follows that $J_k(n)$ is multiplicative.
(c) We already showed that the LHS is multiplicative in (b). The RHS is the product of n^k, which is multiplicative and the other term is also multiplicative. Thus we are reduced to the case when $n = p^e$ is a power of a prime.
In this case the condition that the greatest common divisor is coprime to n is the same as the condition that the greatest common divisor is not divisible by p. The number of k-tuples whose greatest common divisor is divisible by p is

$$(n/p)^k.$$

Thus the number of k-tuples whose greatest common divisor is not divisible by p is

$$\begin{aligned} J_k(p^e) &= n^k - (n/p)^k \\ &= n^k \left(1 - \frac{1}{p^k}\right) \\ &= n^k \prod_{p|n} \left(1 - \frac{1}{p^k}\right). \end{aligned}$$

6.2.9. We have

$$\begin{aligned}\sum_{d_1 d_2=m, e_1 e_2=n} \mu(d_1)\mu(e_1)F(d_2,e_2) &= \sum_{d_1 d_2=m, e_1 e_2=n} \mu(d_1)\mu(e_1) \sum_{d|d_2, e|e_2} f(d,e) \\ &= \sum_{d_1 d|m, e_1 e|n} \mu(d_1)\mu(e_1) f(d,e) \\ &= \sum_{d|m, e|n} f(d,e) \sum_{d_1|m/d, e_1|n/e} \mu(d_1)\mu(e_1) \\ &= \sum_{d|m, e|n} f(d,e) \sum_{d_1|m/d} \mu(d_1) \sum_{e_1|n/e} \mu(e_1) \\ &= \sum_{d|m, e|n} f(d,e) M(m/d)M(n/e) \\ &= f(m,n).\end{aligned}$$

6.3.2. Note that

$$\llcorner 2x \lrcorner - 2\llcorner x \lrcorner$$

is either 0 or 1. On the other hand,

$$\begin{aligned}\llcorner x \lrcorner + \llcorner y \lrcorner + 1 &= \llcorner x + \llcorner y \lrcorner + 1 \lrcorner \\ &\geq \llcorner x + y \lrcorner.\end{aligned}$$

Suppose that

$$\llcorner 2x \lrcorner - 2\llcorner x \lrcorner = 1.$$

We have

$$\begin{aligned}\llcorner 2x \lrcorner + \llcorner 2y \lrcorner &\geq 2\llcorner x \lrcorner + 2\llcorner y \lrcorner + 1 \\ &= \llcorner x \lrcorner + \llcorner y \lrcorner + \llcorner x \lrcorner + \llcorner y \lrcorner + 1 \\ &\geq \llcorner x \lrcorner + \llcorner y \lrcorner + \llcorner x + y \lrcorner.\end{aligned}$$

By symmetry we are also done if

$$\llcorner 2y \lrcorner - 2\llcorner y \lrcorner = 1.$$

Thus we may assume that

$$\llcorner 2x \lrcorner - 2\llcorner x \lrcorner = 0 \qquad \text{and} \qquad \llcorner 2y \lrcorner - 2\llcorner y \lrcorner = 0.$$

Note that if we multiply

$$x = \llcorner x \lrcorner + \{\, x \,\}$$

by 2 we get

$$2x = 2\llcorner x \lrcorner + 2\{\, x \,\}$$

so that

$$\llcorner 2x \lrcorner = 2\llcorner x \lrcorner + \llcorner 2\{\, x \,\} \lrcorner,$$

and so $2\{\, x \,\} < 1$, that is, $\{\, x \,\} < 1/2$.

In this case, since

$$x + y = \llcorner x \lrcorner + \llcorner y \lrcorner + \{ x \} + \{ y \}$$

and $\{ x \} + \{ y \} < 1$, we see that

$$\llcorner x + y \lrcorner = \llcorner x \lrcorner + \llcorner y \lrcorner.$$

Thus

$$\begin{aligned} \llcorner 2x \lrcorner + \llcorner 2y \lrcorner &= 2\llcorner x \lrcorner + 2\llcorner y \lrcorner \\ &= \llcorner x \lrcorner + \llcorner y \lrcorner + \llcorner x \lrcorner + \llcorner y \lrcorner \\ &= \llcorner x \lrcorner + \llcorner y \lrcorner + \llcorner x + y \lrcorner. \end{aligned}$$

6.3.4. (a) One direction is clear. If

$$x = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n},$$

then x is rational, since the rationals are closed under addition.
Now suppose that x is rational. Then we may write

$$x = \frac{p}{q}$$

where p and $q > 0$ are coprime integers. We would like to write down an invariant that goes down at each step.
It is tempting to try to use the denominator. If we start with $x = 2/3$ then $a_1 = 1/2$ and $x_2 = 1/6$ and in this case the denominator increases.
It is tempting to believe that perhaps the denominator always divides $q!$. But if $x = 3/7$ then we get

$$\frac{3}{7} = \frac{1}{3} + \frac{1}{11} + \frac{1}{231},$$

and 231 does not divide 7!. There does not seem to be a simple invariant only depending on the denominator that goes down.
In fact it is easy to see that the numerator always goes down. In fact if we write $q = cp + d$, where $0 \leq d \leq p - 1$ then $c \geq 1$ as $x \leq 1$ implies $q \geq p$. We claim that $a_1 = c + 1$. Indeed

$$\begin{aligned} x - \frac{1}{c} &= \frac{p}{q} - \frac{1}{c} \\ &= \frac{pc - q}{qc} \\ &= \frac{-d}{qc} \\ &< 0, \end{aligned}$$

so that $a_1 \geq c+1$. But

$$\begin{aligned} x - \frac{1}{c+1} &= \frac{p}{q} - \frac{1}{c+1} \\ &= \frac{p(c+1)-q}{q(c+1)} \\ &= \frac{p-d}{q(c+1)} \\ &\geq 0. \end{aligned}$$

Thus $a_1 = c+1$ and

$$x_1 = \frac{p-q}{q(c+1)}.$$

It follows that the numerator of x_1 is at most $p-q<p$.
Thus the algorithm terminates in at most p steps if x has numerator p.
(b) Note that the sum of the reciprocals of $b_1, b_2, \ldots$ converges by assumption. Let

$$x = \sum_{n=1}^{\infty} \frac{1}{b_n}.$$

We show by induction that if we choose the natural numbers $a_1, a_2, \ldots$ as in part (a) then in fact $a_n = b_n$ for all n. Suppose that we have proved this result up to n. Clearly

$$\begin{aligned} x - \sum_{k \leq n} \frac{1}{a_k} &= x - \sum_{k \leq n} \frac{1}{b_k} \\ &= \sum_{k>n} \frac{1}{b_k} \\ &> \frac{1}{b_{n+1}}, \end{aligned}$$

so that $a_{n+1} < b_{n+1}$. On the other hand,

$$\begin{aligned} x - \sum_{k \le n} \frac{1}{a_k} &= x - \sum_{k \le n} \frac{1}{b_k} \\ &= \sum_{k > n} \frac{1}{b_k} \\ &= \frac{1}{b_{n+1}} + \sum_{k > n+1} \frac{1}{b_k} \\ &< \frac{1}{b_{n+1}} + \frac{1}{b_{n+1} - 1} - \frac{1}{b_{n+1}} \\ &= \frac{1}{b_{n+1} - 1}. \end{aligned}$$

Thus $a_{n+1} \ge b_{n+1}$. It follows that $a_{n+1} = b_{n+1}$ and this completes the induction.

As $a_1, a_2, \ldots = b_1, b_2, \ldots$ is an infinite sequence, it follows that the algorithm never terminates and so x is irrational.

Let $b_k = 2^{3^k} + 1$. We have

$$\begin{aligned} \sum_{n \ge k} \frac{1}{b_n} &= \sum_{n \ge k} \frac{1}{2^{3^n} + 1} \\ &\le \sum_{n \ge k} \frac{1}{2^{3^n}} \\ &\le \frac{1}{2^{3^k}} \sum_{n \ge 0} \frac{1}{2^n} \\ &= 2 \frac{1}{2^{3^k}} \\ &\le \frac{1}{2^{3^{(k-1)}}} \\ &= \frac{1}{b_{k-1} - 1}. \end{aligned}$$

Thus

$$\sum (2^{3^k} + 1)^{-1}$$

is irrational.

6.3.6. (a) We prove that

$$\frac{(ab)!}{a!(b!)^a}$$

is a natural number, for all natural numbers a and b. We proceed by induction on a.

If $a = 1$ then

$$\frac{(ab)!}{a!(b!)^a} = \frac{b!}{1!(b!)^1} = 1.$$

Now suppose the result holds for a.

$$\begin{aligned}\frac{((a+1)b)!}{(a+1)!(b!)^{a+1}} &= \frac{(ab)!}{a!(b!)^a}\frac{(a+1)b((a+1)b-1)!}{(ab)!(a+1)b!} \\ &= \frac{(ab)!}{a!(b!)^a}\frac{((a+1)b-1)!}{(ab)!(b-1)!} \\ &= \frac{(ab)!}{a!(b!)^a}\binom{((a+1)b-1)!}{(b-1)!}.\end{aligned}$$

The first term is an integer by induction and the second term is an integer, since it is a binomial.

(b) Pick a prime p. The exponent of the largest power of p dividing $(2a)!$ is

$$\llcorner \frac{2a}{p} \lrcorner + \llcorner \frac{2a}{p^2} \lrcorner + \dots.$$

Thus the exponent of the largest power of p dividing $(2a)!(2b)!$ is

$$\llcorner \frac{2a}{p} \lrcorner + \llcorner \frac{2b}{p} \lrcorner + \llcorner \frac{2a}{p^2} \lrcorner + \llcorner \frac{2b}{p^2} \lrcorner + \dots.$$

On the other hand the exponent of the largest power of p dividing $a!b!(a+b)!$ is

$$\llcorner \frac{a}{p} \lrcorner + \llcorner \frac{b}{p} \lrcorner + \llcorner \frac{a+b}{p} \lrcorner + \llcorner \frac{a}{p^2} \lrcorner + \llcorner \frac{b}{p^2} \lrcorner + \llcorner \frac{a+b}{p^2} \lrcorner + \dots.$$

The difference of the exponent of the numerator and the denominator is then a sum of terms of the form

$$\llcorner \frac{2a}{p^k} \lrcorner + \llcorner \frac{2b}{p^k} \lrcorner - \llcorner \frac{a}{p^k} \lrcorner - \llcorner \frac{b}{p^k} \lrcorner - \llcorner \frac{a+b}{p^k} \lrcorner.$$

By (6.3.2) applied to $\alpha = a/p^k$ and $\beta = b/p^k$ this is non-negative. But then

$$\frac{(2a)!(2b)!}{a!b!(a+b)!}$$

is a natural number, for all natural numbers a and b (since it's denominator is not divisible by any prime).

6.4.2. Note that

$$\sum_{i,j=1:p_i<p_j}^{\infty} \llcorner \frac{x}{p_i p_j} \lrcorner = \sum_{p_i p_j \leq x, p_i<p_j} \llcorner \frac{x}{p_i p_j} \lrcorner$$

$$\leq \sum_{p_i p_j \leq x, p_i<p_j} \frac{x}{p_i p_j} - E,$$

where

$$E \leq \sum_{p_i p_j \leq x, p_i<p_j} 1.$$

The term on the RHS is the number of ways to pick two primes p_i, p_j such that $p_i < p_j$ and $p_i p_j \leq x$. Let $y = p_i p_j$. Then y determines p_i and p_j by unique factorisation and y is a natural number between 1 and x so that $y \leq \llcorner x \lrcorner \leq x$.
Thus E is at most x and so $-E = O(1)$.