MODEL ANSWERS TO THE FIRST HOMEWORK

6.1.1. We have

$$\sigma_k(n) = \sum_{d|n} d^k.$$

Let $f\colon \mathbb{N} \longrightarrow \mathbb{N}$ be the function $f(n) = n^k$. If m and n are are any two natural numbers then

$$\begin{aligned} f(mn) &= (mn)^k \\ &= m^k n^k \\ &= f(m)f(n). \end{aligned}$$

In particular f is multiplicative. Therefore $\sigma_k(n)$ is multiplicative.
Suppose that $n = p^e$ is a power of a prime. Then the divisors of n are the powers of p up to n and so

$$\begin{aligned} \sigma_k(p^e) &= \sum_{i=0}^{e} (p^i)^k \\ &= \sum_{i=0}^{e} p^{ik} \\ &= \sum_{i=0}^{e} (p^k)^i \\ &= \frac{p^{k(e+1)} - 1}{p^k - 1}. \end{aligned}$$

Thus if $n = p_1^{e_1} p_2^{e_2} \dots p_r^{e_r}$ is the prime factorisation then

$$\sigma_k(n) = \prod_{i=1}^{r} \frac{p_i^{k(e_i+1)} - 1}{p_i^k - 1}.$$

6.1.2. We first show that both sides are multiplicative. As τ is a multiplicative function it follows that τ^3 is a multiplicative function and so the LHS is a multiplicative function.
Similarly

$$\sum_{d|n} \tau(d)$$

is multiplicative as τ is multiplicative and so the RHS is multiplicative, as the square of a multiplicative function is multiplicative.

Suppose that $n = p^e$ is a power of a prime. For the LHS we have

$$\begin{aligned}\sum_{d|p^e} \tau^3(d) &= \sum_{i=0}^{e} \tau^3(p^i) \\ &= \sum_{i=0}^{e} (1+i)^3 \\ &= \sum_{i=1}^{e+1} i^3 \\ &= \frac{(e+1)^2(e+2)^2}{4}.\end{aligned}$$

On the other hand, for the RHS we have

$$\begin{aligned}\left(\sum_{d|p^e} \tau(d)\right)^2 &= \left(\sum_{i=0}^{e} \tau(p^i)\right)^2 \\ &= \left(\sum_{i=0}^{e} (1+i)\right)^2 \\ &= \left(\sum_{i=1}^{e+1} i\right)^2 \\ &= \left(\frac{(e+1)(e+2)}{2}\right)^2.\end{aligned}$$

Thus we have equality when n is a power of a prime.
Suppose that $p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$ is the prime factorisation of n. Suppose the LHS is the function $l(n)$ and the RHS is the function $r(n)$. We have

$$\begin{aligned}l(n) &= l(p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}) \\ &= l(p_1^{e_1}) l(p_2^{e_2}) \dots l(p_k^{e_k}) \\ &= r(p_1^{e_1}) r(p_2^{e_2}) \dots r(p_k^{e_k}) \\ &= r(p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}) \\ &= r(n).\end{aligned}$$

6.1.3. Suppose that $n = p^e$ is a prime power. Then

$$\sigma(n) = 1 + p + p^2 + \cdots + p^e.$$

Hence $\sigma(n)$ is odd if and only if

$$p + p^2 + \cdots + p^e$$

is even. If $p = 2$ then $\sigma(n)$ is always odd and p is odd then e is even.

If $n = 2^{e_0} p_1^{e_1} p_2^{e_2} \dots p_r^{e_r}$ is the prime factorisation of n then $\sigma(p_i^{e_i})$ is odd by multiplicativity so that we must have e_i is even for all $0 < i \leq r$. If e_0 is even it follows that n is a square and if e_0 is odd then n is twice a square.

6.1.5. If we expand the RHS we get

$$\begin{aligned}
\left(\sum_{n=1}^{\infty} \frac{1}{n^s}\right)^2 &= \left(\sum_{m=1}^{\infty} \frac{1}{m^s}\right)\left(\sum_{n=1}^{\infty} \frac{1}{n^s}\right) \\
&= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^s} \frac{1}{n^s} \\
&= \sum_{m=1}^{\infty} \sum_{d=1}^{\infty} \frac{1}{(md)^s} \\
&= \sum_{n=1}^{\infty} (\sum_{d|n} \frac{1}{n^s}) \\
&= \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s}.
\end{aligned}$$

Here we used the fact that since the series converges and all terms are positive, it follows that the series converges absolutely, so that we are free to rearrange the terms of the sum.

6.1.6. (a) Let d_1 be an odd divisor of n. Suppose that $n = 2^e m$ where m is odd. We have

$$\begin{aligned}
\sum_{i=0}^{e} (-1)^{n/2^i d_1} 2^i d_1 &= (-1)^{n/d_1} d_1 (1 + 2 + 2^2 + \cdots + 2^{e-1} - 2^e) \\
&= (-1)^{n/d_1} d_1 \left(\frac{2^e - 1}{2 - 1} - 2^e\right) \\
&= -(-1)^{n/d_1} d_1.
\end{aligned}$$

Thus

$$\begin{aligned}
-\sum_{d|n} (-1)^{n/d} d &= \sum_{d_1|m} - \sum_{i=0}^{e} (-1)^{n/2^i d_1} 2^i d_1 \\
&= \sum_{d_1|m} d_1.
\end{aligned}$$

(b) Suppose that n is even, so that $n = 2l$ for some natural number l and $e > 0$.

$$\begin{aligned}2\sigma(l) - \sigma(n) &= \sum_{d|l} 2d - \sum_{d|n} d \\ &= \sum_{2d|l} d - \sum_{d|n} d \\ &= \sum_{d_1|m} d_1 \\ &= \sum_{d|n} (-1)^{n/d} d.\end{aligned}$$

6.1.8. Suppose that m and n are coprime. If m is divisible by μ distinct primes and n is divisible by ν distinct primes then mn is divisible by $\mu + \nu$ distinct primes, so that

$$\omega(mn) = \omega(m) + \omega(n)$$

and so ω is additive.

6.1.9. If $n = p_1^{e_1} p_2^{e_2} \dots p_r^{e_r}$ is the prime factorisation of n and $p_1 \leq p_2 \leq p_3 \leq \cdots \leq p_r$ then clearly

$$p_k \geq k + 1.$$

Note also that $r = \omega(n)$. If $n = p^e$ is a power of a prime and $p \geq k$ then

$$\begin{aligned}\varphi(n) &= p^e - p^{e-1} \\ &= n - \frac{n}{p} \\ &= n(1 - \frac{1}{p}) \\ &\geq n(1 - \frac{1}{k}).\end{aligned}$$

Thus

$$\begin{aligned}
\varphi(n) &= \varphi(p_1^{e_1} p_2^{e_2} \dots p_r^{e_r}) \\
&= \varphi(p_1^{e_1})\varphi(p_2^{e_2}) \dots \varphi(p_r^{e_r}) \\
&\geq \prod_{k=1}^{\omega(n)} p^{e_i}(1 - \frac{1}{k+1}) \\
&= n \prod_{k=2}^{\omega(n)+1} (1 - \frac{1}{k}) \\
&= n \prod_{k=2}^{\omega(n)+1} \frac{k-1}{k} \\
&= \frac{n}{\omega(n)+1}.
\end{aligned}$$

To establish the inequalities

$$2^{\omega(n)} \leq \tau(n) \leq n,$$

note that all three terms are positive and multiplicative. So we may assume that $n = p^e$ is a power of a prime. In this case $\omega(n) = 1$, unless $e = 0$ so that $n = 1$ and $\tau(n) = 1 + e$ and the inequalities are clear.
Taking logs we see that

$$2^{\omega(n)} \leq n \qquad \text{implies that} \qquad \omega(n) \leq \frac{\log n}{\log 2}.$$

Note that $\omega(n) + 1 \leq 2\omega(n)$ and so

$$\varphi(n) \geq \frac{\log 2}{2 \log n} n.$$

6.2.1. We have that

$$\sigma(n) = \sum_{d|n} d$$

so that this result follows by Möbius inversion.
6.2.2. We may write n in the form $n = n_1^2 n_2$ where n_2 is square-free. Suppose that $d^2|n$. Suppose that p is a prime and p^e divides d. Then p^{2e} divides n and so p^{2e-1} divides n_1^2, since p^2 does not divide n_2. But then p^e divides n_1. Thus d divides n_1.
In this case $d^2|n$ if and only if $d|n_1$. It follows that

$$\sum_{d^2|n} \mu(d) = \sum_{d|n_1} \mu(d).$$

But

$$\sum_{d|n_1} \mu(d) = \begin{cases} 1 & \text{if } n_1 = 1 \\ 0 & \text{otherwise.} \end{cases}$$

But $\mu(n)$ is zero if and only if n is square-free. Thus

$$|\mu(n)| = \begin{cases} 1 & \text{if } n_1 = 1 \\ 0 & \text{otherwise.} \end{cases}$$