SECOND MIDTERM
MATH 104B, UCSD, WINTER 18

You have 80 minutes.

There are 5 problems, and the total number of points is 70. Show all your work. *Please make your work as clear and easy to follow as possible.*

Name:________________________________

Signature:________________________________

Student ID #:________________________________

Problem	Points	Score
1	15	
2	10	
3	15	
4	20	
5	10	
6	10	
7	10	
Total	70	

1. (15pts) (i) *Give the definition of* $\mathrm{li}(x)$.

$$\mathrm{li}(x) = \int_2^x \frac{1}{\log t}\,\mathrm{d}t.$$

(ii) *Give the definition of* $\vartheta(x)$.

$$\vartheta(x) = \sum_{p \le x} \log p.$$

(iii) *Give the definition of the Riemann zeta-function* $\zeta(s)$.

$$\zeta(s) = \sum_k \frac{1}{k^s}.$$

2. (10pts) *Show that*

$$\mathrm{li}(x) = \frac{x}{\log x} + \frac{1!x}{\log^2 x} + \frac{2!x}{\log^3 x} + \cdots + \frac{(n-1)!x}{\log^n x} + O\left(\frac{x}{\log^{n+1} x}\right).$$

Note that if we integrate by parts then we get

$$\begin{aligned}\int_0^x \frac{\mathrm{d}t}{\log^n t} &= \int_0^x 1 \cdot \frac{\mathrm{d}t}{\log^n t} \\ &= \left[\frac{t}{\log^n t}\right]_0^x + n \int_0^x \frac{t\,\mathrm{d}t}{t \log^{n+1} t} \\ &= \frac{x}{\log^n x} + n \int_0^x \frac{\mathrm{d}t}{\log^{n+1} t}.\end{aligned}$$

It follows by induction that

$$\mathrm{li}(x) = \frac{x}{\log x} + \frac{1!x}{\log^2 x} + \frac{2!x}{\log^3 x} + \cdots + \frac{(n-1)!x}{\log^n x} + n! \int_0^x \frac{\mathrm{d}t}{\log^{n+1} t}.$$

Now to estimate the last integral, we break it into three parts.

$$\int_0^x \frac{\mathrm{d}t}{\log^{n+1} t} = \int_0^2 \frac{\mathrm{d}t}{\log^{n+1} t} + \int_2^{\sqrt{x}} \frac{\mathrm{d}t}{\log^{n+1} t} + \int_{\sqrt{x}}^x \frac{\mathrm{d}t}{\log^{n+1} t}.$$

The first integral is constant. The second is over an interval of length bounded by $\sqrt{x}$ of a function bounded by a constant ($\frac{1}{\log^{n+1} 2}$) and so the second integral is $O(\sqrt{x})$. The third integral is over an interval of length bounded by x of a function which is bounded by

$$\frac{1}{\log^{n+1} \sqrt{x}} = O\left(\frac{1}{\log^{n+1} x}\right).$$

Thus the last integral is

$$O\left(\frac{x}{\log^{n+1} x}\right).$$

Therefore

$$\left| n! \int_0^x \frac{\mathrm{d}t}{\log^{n+1} t} \right| = O\left(\frac{x}{\log^{n+1} x}\right).$$

and so the result follows.

3. (15pts) Show that
(i)

$$\pi(x) \leq r + x\prod_{i=1}^{r}\left(1-\frac{1}{p_i}\right) + 2^r,$$

where $p_1, p_2, \dots, p_r$ are the first r primes.

Let

$$P = \{\, n \in \mathbb{N} \,|\, 1 < n \leq x \text{ and } n \text{ is a not a multiple of } p_1, p_2, \dots, p_r \,\},$$

so that P is the set of integers from 2 to x which are not multiples of $p_1, p_2, \dots, p_r$. Let $A(x, r)$ be the cardinality of P.
If p is a prime from 1 to n then either p is one of $p_1, p_2, \dots, p_r$ or p belongs to P. It follows that

$$\pi(x) \leq r + A(x, r).$$

We want to estimate $A(x, r)$. Let M_i be the set of integers from 1 to n which are multiples of p_i. Let M_{ij} be the set of integers from 1 to n which are multiples of both p_i and p_j. As p_i and p_j are coprime,

$$M_{ij} = M_i \cap M_j.$$

Note that

$$|M_i| = \llcorner\frac{x}{p_i}\lrcorner \qquad \text{and} \qquad |M_{ij}| = \llcorner\frac{x}{p_i p_j}\lrcorner,$$

and so on. It follows by inclusion-exclusion that

$$A(x, r) = \llcorner x \lrcorner - \sum_{i=1}^{r}\llcorner\frac{x}{p_i}\lrcorner + \sum_{i \neq j \leq r}\llcorner\frac{x}{p_i p_j}\lrcorner + \dots + (-1)^r\llcorner\frac{x}{p_1 p_2 \dots p_r}\lrcorner.$$

Suppose that we approximate the RHS by simply ignoring all of the round downs,

$$x - \sum_{i=1}^{r}\frac{x}{p_i} + \sum_{i \neq j \leq r}\frac{x}{p_i p_j} + \dots + (-1)^r\frac{x}{p_1 p_2 \dots p_r}.$$

The worse case scenario for the error is

$$1 + \binom{r}{1} + \binom{r}{2} + \dots + \binom{r}{r} = 2^r.$$

It follows that

$$\pi(x) \leq r + x\prod_{i=1}^{r}\left(1-\frac{1}{p_i}\right) + 2^r.$$

(b) If $x \geq 2$ then

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) < \frac{1}{\log x}.$$

We compute the product of the reciprocals,

$$\prod_{p \leq x} \frac{1}{1 - \frac{1}{p}} = \prod_{p \leq x} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \dots\right).$$

Consider what happens if we expand the RHS. If m is an integer which is a product of primes less than x then the term $\frac{1}{m}$ appears somewhere in the expansion of this product.

Now any integer $m \leq x$ is a product of primes less than x and so

$$\begin{aligned}\prod_{p \leq x} \frac{1}{1 - \frac{1}{p}} &> \sum_{k=1}^{n} \frac{1}{k} \\ &> \int_{1}^{\ulcorner x \urcorner} \frac{\mathrm{d}u}{u} \\ &> \log x.\end{aligned}$$

(c)

$$\pi(x) \ll \frac{x}{\log\log x}.$$

$$\begin{aligned}
\pi(x) &\le r + x\prod_{i=1}^{r}\left(1-\frac{1}{p_i}\right) + 2^r && \text{as proved above}\\
&\le 2^{r+1} + x\prod_{i=1}^{r}\left(1-\frac{1}{p_i}\right) && \text{as } r \le 2^r\\
&\le 2^{r+1} + \frac{x}{\log p_r} && \text{using (b)}\\
&\le 2^{r+1} + \frac{x}{\log r} && \text{as } p_r \ge r\\
&\le 2^{\log x+2} + \frac{x}{\log\log x} && \text{take } r = \ulcorner \log x \urcorner\\
&\le 4\cdot 2^{\log x} + \frac{x}{\log\log x}\\
&= O\left(2^{\log x}\right) + \frac{x}{\log\log x}\\
&\le o\left(\frac{x}{\log\log x}\right) + \frac{x}{\log\log x} && \text{as } \log 2 < 1\\
&= O\left(\frac{x}{\log\log x}\right).
\end{aligned}$$

4. (20pts) *Let* $\langle x \rangle = \llcorner x + 1/2 \lrcorner$ *denote the nearest integer to* x.
(i) *if* x *is a real number then show that*

$$\llcorner x \lrcorner = \langle x/2 \rangle + \llcorner x/2 \lrcorner.$$

Suppose that $\{\, x/2 \,\} < 1/2$. Then

$$\langle x/2 \rangle = \llcorner x/2 \lrcorner \qquad \text{and} \qquad \llcorner x \lrcorner = 2\llcorner x/2 \lrcorner$$

and so

$$\begin{aligned} \llcorner x \lrcorner &= 2\llcorner x/2 \lrcorner \\ &= \llcorner x/2 \lrcorner + \langle x/2 \rangle. \end{aligned}$$

Now supppose that $\{\, x/2 \,\} \geq 1/2$. Then

$$\langle x/2 \rangle = \llcorner x/2 \lrcorner + 1 \qquad \text{and} \qquad \llcorner x \lrcorner = 2\llcorner x/2 \lrcorner + 1$$

and so

$$\begin{aligned} \llcorner x \lrcorner &= 2\llcorner x/2 \lrcorner + 1 \\ &= \llcorner x/2 \lrcorner + \langle x/2 \rangle. \end{aligned}$$

(ii) *Let* $p_1, p_2, \ldots, p_m$ *be the first* m *odd primes and let* $P(x, m)$ *be the number* of odd integers at most x and not divisible by any of these primes.
Show that

$$P(x, m) = \sum_a \langle x/2a \rangle - \sum_b \langle x/2b \rangle$$

where a and b run over all products of an even and an odd number of primes among $p_1, p_2, \ldots, p_m$ respectively.

$$\begin{aligned} P(x, m) &= A(x, m+1) + 1 \\ &= \llcorner x \lrcorner - \llcorner x/2 \lrcorner - \left(\sum \llcorner x/p_i \lrcorner - \sum \llcorner x/2p_i \lrcorner\right) + \left(\sum \llcorner x/p_i p_j \lrcorner - \sum \llcorner x/2p_i p_j \lrcorner\right) + \ldots \\ &= \langle x/2 \rangle - \sum \langle x/p_i \rangle + \sum \langle x/p_i p_j \rangle + \ldots \\ &= \sum_a \langle x/2a \rangle - \sum_b \langle x/2b \rangle. \end{aligned}$$

(iii) *Show that*

$$\pi(x) = \pi(\sqrt{x}) + P(x, \pi(\sqrt{x}) - 1) - 1.$$

Let $r = \pi(\sqrt{x})$. Then

$$\begin{aligned} \pi(x) &= r + A(x, r) \\ &= \pi(\sqrt{x}) + P(x, \pi(\sqrt{x}) - 1. \end{aligned}$$

(iv) Use (iii) *to calculate* $\pi(200)$.

Now the odd primes up to 14 are 3, 5, 7, 11 and 13. Thus

$$\pi(\sqrt{200}) = 6.$$

On the other hand, one can compute

$$\begin{aligned} P(200, 5) &= 100 - (33 + 20 + 14 + 9 + 8) + (7 + 5 + 3 + 3 + 3 + 2 + 2 + 1 + 1 + 1) - (1 + 1 + 1 \\ &= 41. \end{aligned}$$

Thus

$$\begin{aligned} \pi(200) &= \pi(14) + P(200, 5) - 1 \\ &= 6 + 41 - 1 \\ &= 46. \end{aligned}$$

5. (10pts) *Derive the prime number theorem from the relation*

$$\vartheta(x) \sim x.$$

We have

$$\begin{aligned} x &\sim \sum_{p \le x} \log p \\ &\le \sum_{p \le x} \log x \\ &= \log x \sum_{p \le x} 1 \\ &= \pi(x) \log x. \end{aligned}$$

On the other hand

$$\begin{aligned} x &\sim \sum_{x^{1-\epsilon} \le p \le x} \log p \\ &\ge \sum_{x^{1-\epsilon} \le p \le x} \log x^{1-\epsilon} \\ &= (1-\epsilon) \log x \sum_{x^{1-\epsilon} \le p \le x} 1 \\ &= (1-\epsilon)(\pi(x) - \pi(x^{1-\epsilon})) \log x \\ &= (1-\epsilon)(\pi(x) + O(x^{1-\epsilon})) \log x. \end{aligned}$$

Putting these together, we see that

$$\pi(x) \sim \frac{x}{\log x}.$$

Bonus Challenge Problems

6. (10pts) *Show that*

$$\int_2^x \frac{\pi(t)}{t^2}\,\mathrm{d}t = \sum_{p\le x}\frac{1}{p} + o(1).$$

We apply partial summation to

$$\lambda_n = p_n \qquad c_n = 1 \qquad \text{and} \qquad f(x) = \frac{1}{x}.$$

We get

$$\sum_{p\le x}\frac{1}{p} = \frac{\pi(x)}{x} - \int_2^x -\frac{\pi(t)}{t^2}\,\mathrm{d}t.$$

As

$$\pi(x) = O\left(\frac{x}{\log\log x}\right)$$

the first expression on the RHS is certainly $o(1)$. Rearranging we get

$$\int_2^x \frac{\pi(t)}{t^2}\,\mathrm{d}t = \sum_{p\le x}\frac{1}{p} + o(1).$$

7. (10pts) *Show that there are constants* c_1 *and* c_2 *such that*

$$c_1 \frac{x}{\log x} < \pi(x) < c_2 \frac{x}{\log x}.$$

See lecture 7.